V.S. Pilidi

ANALYTIC GEOMETRY

textbook

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The book contains material on analytic geometry included in the university discipline "Algebra and Geometry". In addition to detailed presentation of theoretical material, there are given problems in the volume that is quite sufficient both for practical classes and for students' independent work. Most problems are provided with detailed solutions. The book is addressed to students of the educational program "Theoretical Computer Science and Information Technologies" and can also be used by students of other educational programs.

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Chapter 1

Straight lines on the plane

1.1 Coordinates on the plane

1.1.1 Definition of coordinates

The place of a point M on a straight line is fully determined by its distance OM from a fixed point O on the line, if we know on which side of the point O is the point M (right or left), see Fig. 1.

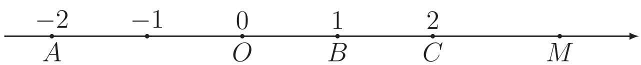

Fig. 1. Coordinate axis.

It is assumed that on this line there is used some unit in which the distances are measured. The fixed point O is called the *origin*. The distance OM of the point $M \neq O$, taken with the sign “plus” if M lies to the right of the origin and with the sign “minus” when M lies to the left from O, is called the *coordinate* of M.The coordinate of the point O is set to zero. In this case, an arrow is added to the line indicating the positive direction on it.

Let us select, on a given line, an arbitrary origin O, a unit of measure, and a definite positive direction. Then any real number, regarded as the coordinate of a point M, fully determines the position of M on this line. And conversely, every point on the line has one and only one coordinate. In general case the coordinate of a point is usually denoted by the letter x, which, as we told above, may be any real number. In this case we write $M(x)$. For instance, $A(-2)$, $B(1)$, $C(2)$ (Fig. 1).

Such a line is called the *coordinate axis.* The distance between the points $M_1(x_1)$ and $M_2(x_2)$ on the coordinate axis is found by the formula $|M_1M_2| = |x_2 - x_1|$. The midpoint of the segment M_1M_2 is the point $M\left(\frac{x_1+x_2}{2}\right)$.

To locate a point on the plane that is, to determine its position, there is used the following well known approach. We suppose that there is defined a unit to measure distances on the plane. We draw two lines at right angles on the plane, the point of intersection of these lines is called the *origin* and is usually denoted by the letter O. Then we define positive directions on each of the lines denoting these directions by arrows (see Fig. 2, the points on the axes match unit steps).

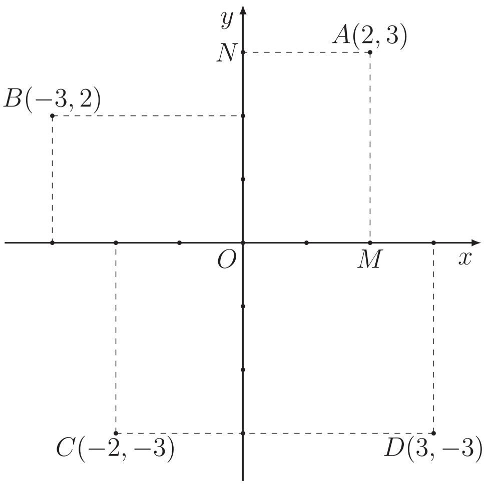

Fig. 2. Coordinates on the plane

We emphasize that on the both lines there is an origin, a unit of measure and a positive direction. Therefore, there are defined coordinates on the both lines. These two lines are called the *axes of coordinates.* To distinguish these lines one of them is called the axis Ox, x-axis, or axis of *abscissas*, and the other is called the axis Oy, the y-axis, or axis of *ordinates.*

Now we take an arbitrary point A on the plane, and project it on each axis, i.e. we drop the perpendiculars AM and AN from A on the axes. The coordinate x of the point M on the x-axis is called the *abscissa* of A. The coordinate y of the point N on the y-axis is called the *ordinate* of A. The position of the point A on the plane is fully determined if its abscissa x and its ordinate y are both given. The two numbers x, y are also called the *coordinates* of the point A. This fact is denoted as follows: $A(x, y)$. In some cases there is used the notation (x, y) without denoting the point itself. For example, it is possible to say: "let us take the points $(1, 2)$ and $(-4, 3)$".

As a consequence of the correspondence given above, we can state that *every point on the plane has two definite real numbers as coordinates; conversely, every pair of real numbers defines one and only one point of the plane.*

The plane equipped with the coordinate system is called the *coordinate plane.* The coordinate system we are using is called the *Cartesian* (or *rectangular*) coordinate system. The concept of Cartesian coordinates may be generalized to the case when the axes are not necessarily perpendicular to each other, and there are different units along the axes. In the following we don't use such a generalization.

The axes divide the plane into four parts which are called the first, second, third and fourth *quadrants* (Fig. 3). Quadrants are also numbered with Arabic numerals (1, 2, 3, 4) or Roman numerals (I, II, III, IV) and the numbering is counterclockwise. The inequalities defining the quadrants are given below.

Quadrant I: $x > 0,\ y > 0$; quadrant II: $x < 0,\ y > 0$;

quadrant III: $x < 0$, $y < 0$; quadrant IV: $x > 0$, $y < 0$.

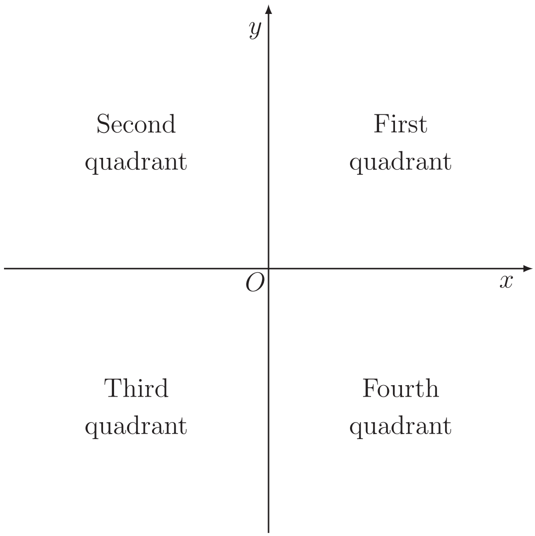

Fig. 3. Quadrants on the plane

1.1.2 Distance between two points

Let us consider two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ on the plane. To find the distance d between these points, we drop perpendiculars M_1P_1, M_2P_2 on the x-axis and perpendiculars M_1Q_1, M_2Q_2 on the y-axis (Fig. 4).

We get the right triangle ΔM_1M_2R with the legs M_1R, M_2R and the hypotenuse M_1M_2. From the equalities

$$M_1R = P_1P_2 = |x_2 - x_1|, \quad M_2R = Q_1Q_2 = |y_2 - y_1|,$$

and the Pythagorean theorem we get that

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The distance d between the points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ is obtained by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

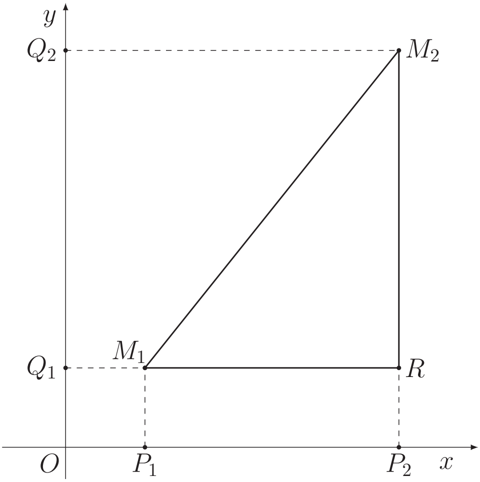

Fig. 4. Distance between two points

1.1.3 Midpoint of a Segment

Let us take two points $A(a_1, a_2)$ and $C(c_1, c_2)$ on the coordinate plane. Coordinates of the midpoint B of the segment AC are the arithmetic means of the corresponding coordinates of A and C, that is, if $B(b_1, b_2)$ then

$$b_1 = \frac{a_1 + c_1}{2}, \quad b_2 = \frac{a_2 + c_2}{2}.$$

To prove the first relation we drop the perpendiculars AP, BQ and CR on the x-axis (see Fig. 5). Then the point Q becomes the midpoint of the segment PR. On the x-axis the points P and R have the following coordinates: $P(a_1)$ and $R(c_1)$. Therefore the point Q has the coordinate $\frac{a_1+c_1}{2}$, and this value is the first coordinate of the point B. Similarly we get that the second coordinate of the point B is $\frac{a_2+c_2}{2}$, so $B\left(\frac{a_1+c_1}{2}, \frac{a_2+c_2}{2}\right)$.

The midpoint of the segment AC where $A(a_1, a_2)$, $C(c_1, c_2)$ is the point $B\left(\frac{a_1+c_1}{2}, \frac{a_2+c_2}{2}\right)$.

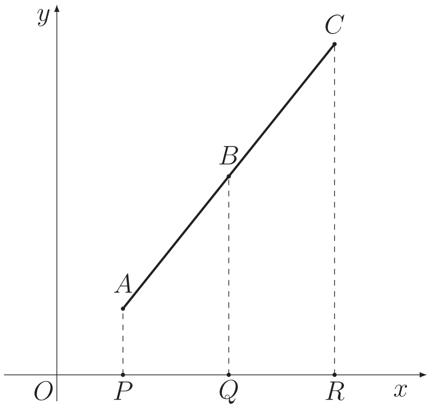

Fig. 5. Midpoint of a segment

1.1.4 Vectors on the coordinate plane

A *vector* is a directed segment of a line, that is, a segment for which it is indicated which of its boundary points is the beginning point and which is the ending point. The vector with the beginning point A and the ending point B is usually denoted by $\overrightarrow{AB}$. The points A and B are also called the *tail* and *head* of the vector $\overrightarrow{AB}$, respectively. Vectors are sometimes indicated by small Latin letters with the arrow (sometimes with the dash) above them, for example $\vec{a}$ or $\bar{a}$. Another common use is to mark vectors in bold characters: $\mathbf{a}$. The length $|\overrightarrow{AB}|$ of the vector $\overrightarrow{AB}$ is defined by the equality $|\overrightarrow{AB}| = |AB|$, i.e. it is the length of the segment AB. In the special case when the points A and B coincide then the vector $\overrightarrow{AB} = \overrightarrow{AA}$ is denoted by $\vec{0}$ (or $\bar{0}$ and $\mathbf{0}$) and is called the *zero vector.* The length of the zero vector equals zero and its direction is assumed to be undetermined. It is assumed that the vector may be displaced parallel to itself. It means that if the vectors $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ have equal lengths, i.e. $|\overrightarrow{AB}| = |\overrightarrow{A_1B_1}|$ and the same directions they are considered to be equal (see Fig. 6).

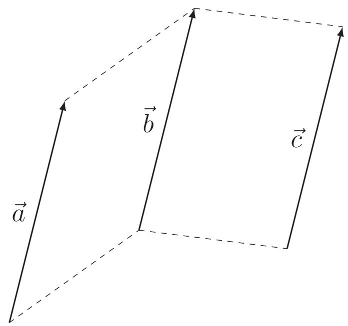

Fig. 6. Equal vectors: $\vec{a} = \vec{b} = \vec{c}$

Vectors lying on parallel straight lines are called *collinear*[1]. The zero vector is considered to be collinear with any vector.

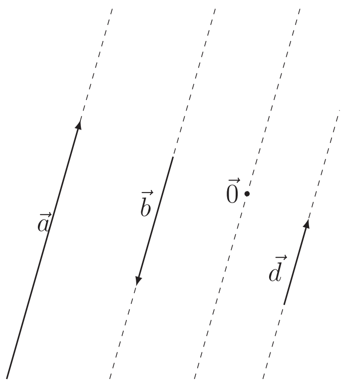

Fig. 7. Collinear vectors

Remark. It should be noted that the binary relation "two vectors are collinear" in the case of vectors on the plane (and in the case of vectors in the space considered below) is not *transitive*. It means that collinearity of vectors $\vec{a}$ and $\vec{b}$ and collinearity of vectors $\vec{b}$ and $\vec{c}$ do not necessarily

[1] We use the definition of parallel lines, which is slightly different from the "school" one. We assume that two straight lines are parallel if they are equal, or if they have no common points. Such a definition is also used in mathematical texts. In our case it somewhat facilitates the statements about collinear vectors and parallel straight lines.

imply collinearity of vectors $\vec{a}$ and $\vec{c}$. For example for any vectors $\vec{a}$ and $\vec{c}$ the pairs $\vec{a}$ and $\vec{0}$, $\vec{0}$ and $\vec{c}$ are collinear, while $\vec{a}$ and $\vec{c}$ are not necessarily collinear.

Now we introduce operation with vectors.

Multiplication by a scalar. For a vector $\vec{a}$ and a number λ, the product $\vec{b} = \lambda\vec{a}$ is uniquely determined by the following properties: $|\vec{b}| = |\lambda|\,|\vec{a}|$, if $\vec{a} \neq \vec{0}$ then $\vec{b}$ has the same direction as $\vec{a}$ if $\lambda > 0$ and the opposite direction if $\lambda < 0$, if $\vec{a} = \vec{0}$ or $\lambda = 0$ then from the equality given above we get that $\vec{b} = \vec{0}$ and the question of its direction does not arise.

The numbers that vectors are multiplied by are usually called *scalars*.

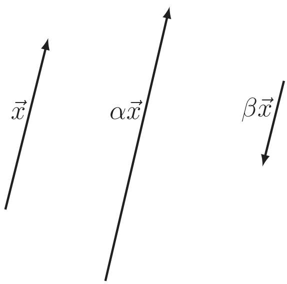

Fig. 8. Multiplication of a vector by a scalar: $\alpha > 0$, $\beta < 0$

Addition. Addition of vectors is performed according to the rule indicated in the following figure. This rule is called the *parallelogram* (or the *triangle*) rule. The meaning of these words is clear from the following figure.

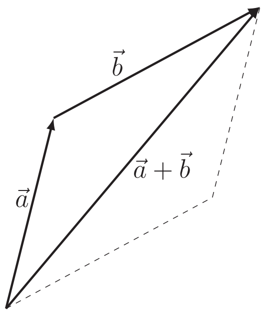

Fig. 9. Sum of vectors

Subtraction. The difference of the vectors $\vec{a}$ and $\vec{b}$ is the single vector $\vec{c} = \vec{a} - \vec{b}$ such that $\vec{c} + \vec{b} = \vec{a}$.

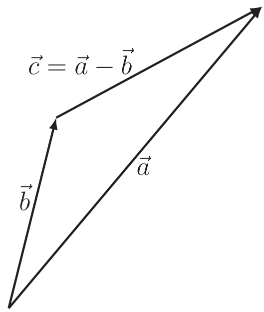

Fig. 10. Difference of vectors

In the given below definition of scalar product and in some other cases we use such an agreement.

> *A product containing probably undetermined factors is set to be zero if at least one of the factors being the part of this product takes the zero value.*

Scalar product. For any *nonzero* vectors $\vec{a}$ and $\vec{b}$ there is defined the angle φ between them satisfying the condition $0 \leqslant \varphi \leqslant \pi$. To find this angle we use parallel translation and reduce the vectors to the common initial point. If one of these vectors takes the zero value then this angle is considered undetermined. The *scalar product* of these vectors is the *number* denoted by $\vec{a}\,\vec{b}$ which equals the product $|\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$. Sometimes the scalar product is also denoted as $(\vec{a}, \vec{b})$ or $\vec{a} \cdot \vec{b}$ and is called the *dot product.*

Coordinates of a vector. For the points $A(a_1, a_2)$ and $B(b_1, b_2)$ on the coordinate plane the *coordinates* of the vector $\overrightarrow{AB}$ are assumed to be equal $b_1 - a_1$ and $b_2 - a_2$ respectively. This fact is written in the following form: $\overrightarrow{AB} = \{b_1 - a_1, b_2 - a_2\}$. The operations with the vectors introduced above may be written in the coordinate form as follows: if λ is a number,

$\vec{a} = \{a_1, a_2\}$, $\vec{b} = \{b_1, b_2\}$, then

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}, \quad \lambda\vec{a} = \{\lambda a_1, \lambda a_2\},$$
$$\vec{a} + \vec{b} = \{a_1 + b_1, a_2 + b_2\}, \quad \vec{a} - \vec{b} = \{a_1 - b_1, a_2 - b_2\}.$$

In order to find conditions of collinearity in the coordinate form, we introduce the following additional agreement: the equality

$$\frac{A}{B} = \frac{C}{D} \tag{1}$$

means that $AD = BC$. Being the standard fact in the case when the denominators do not vanish, such an agreement makes it possible to consider the expressions of the form (1) in the cases when $B = 0$ or $D = 0$. For instance the relations

$$\frac{1}{0} = \frac{2}{0}, \quad \frac{1}{0} = \frac{0}{0}$$

are valid from this point of view. Of course, "fractions" of this kind define no numerical values. Such an agreement allows to give the collinearity condition in an easy-to-remember form.

Remark. For such "fractions" the usual transitivity property of the equality relation may be violated. For instance we have the following equalities understood in the sense indicated above

$$\frac{1}{2} = \frac{0}{0}, \quad \frac{0}{0} = \frac{1}{3},$$

but, of course $\frac{1}{2} \neq \frac{1}{3}$. That is why, speaking below (page 20) about the conditions when two equations define the same straight line, it is said that the relations (7) are equivalent to *three* equalities.

We consider two vectors $\vec{a} = \{a_1, a_2\}$ and $\vec{b} = \{b_1, b_2\}$ on the coordinate plane. These vectors are collinear if and only if at least one of the following relations is valid: $\vec{b} = \lambda\vec{a}$ for some scalar λ, $\vec{a} = \mu\vec{b}$ for some scalar μ (if $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ then these properties take place simultaneously). Hence we get that *the vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if*

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

For instance for any vector $\vec{x} = \{x_1, x_2\}$ and the zero vector $\vec{0} = \{0, 0\}$ we have the relation which is valid from the "extended" point of view:

$$\frac{x_1}{0} = \frac{x_2}{0}.$$

Using the notion of second order determinant, the condition of collinearity may be given in the following form: *the vectors $\vec{a} = \{a_1, a_2\}$ and $\vec{b} = \{b_1, b_2\}$ are collinear if and only if*

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0.$$

Now we are going to express the scalar product in the coordinate form. First we remind some well known fact from the school math course.

The law of cosines

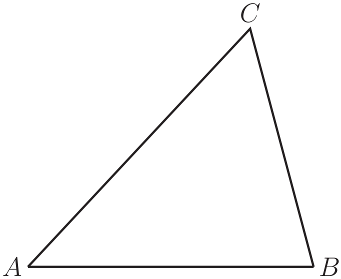

Fig. 11. *For the triangle ΔABC the following equality is valid:*

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB| \cdot |AC| \cdot \cos A.$$

THEOREM 1. *For the vectors $\vec{a} = \{a_1, a_2\}$ and $\vec{b} = \{b_1, b_2\}$ on the coordinate plane the following equality holds $\vec{a}\,\vec{b} = a_1 b_1 + a_2 b_2$.*

PROOF. If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then the formula being proved is obviously true, since both sides take zero value.

Now we turn to the case $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$. Let φ be the angle between these vectors. We will consider two cases, whether these vectors are collinear or not.

1) Assume that that the vectors $\vec{a}$ and $\vec{b}$ are not collinear. In this case $0 < \varphi < \pi$.

Considering the triangle defined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{a}-\vec{b}$ (see Fig. 12) and applying the law of cosines we get:

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\underbrace{|\vec{a}|\,|\vec{b}|\,\cos\varphi}_{\vec{a}\,\vec{b}},$$

whence

$$\vec{a}\,\vec{b} = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a}-\vec{b}|^2}{2}. \tag{2}$$

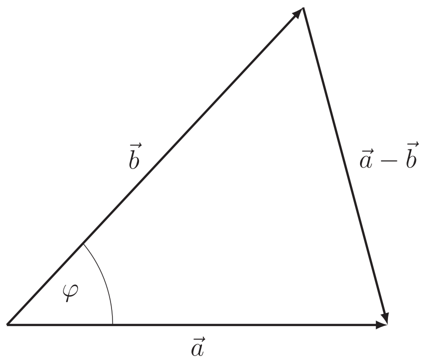

Fig. 12.

Simplifying the numerator of the last fraction we get:

$$\begin{aligned}|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a}-\vec{b}|^2 &= a_1^2 + a_2^2 + b_1^2 + b_2^2 - (a_1-b_1)^2 - (a_2-b_2)^2\\ &= a_1^2 + a_2^2 + b_1^2 + b_2^2 - a_1^2 + 2a_1b_1 - b_1^2 - a_2^2 + 2a_2b_2 - b_2^2\\ &= 2(a_1b_1 + a_2b_2).\end{aligned}$$

Hence from (2) we get: $\vec{a}\,\vec{b} = a_1b_1 + a_2b_2$.

2) Assume now that *nonzero* vectors $\vec{a}$ and $\vec{b}$ are collinear. Then $\varphi = 0$ or $\varphi = \pi$. First we will find the value of the product $|\vec{b}|\cos\varphi$.

If $\varphi = 0$ then $\vec{b} = \lambda\vec{a}$ for some $\lambda > 0$, $|\vec{b}| = \lambda|\vec{a}|$, $\cos\varphi = 1$, and

$$|\vec{b}|\cos\varphi = \lambda|\vec{a}|. \tag{3}$$

If $\varphi = \pi$ then $\vec{b} = \lambda\vec{a}$ for some $\lambda < 0$, $|\vec{b}| = |\lambda| \cdot |\vec{a}| = -\lambda|\vec{a}|$, $\cos\varphi = -1$ and we again get the equality (3).

From (3) we obtain

$$\vec{a}\,\vec{b} = |\vec{a}|\,\underbrace{|\vec{b}|\,\cos\varphi}_{\lambda|\vec{a}|} = \lambda|\vec{a}|^2 = \lambda(a_1^2 + a_2^2)$$
$$= a_1\underbrace{\lambda a_1}_{b_1} + a_2\underbrace{\lambda a_2}_{b_2} = a_1b_1 + a_2b_2.$$

In this case the formula under consideration is also valid. □

Corollary. *The scalar multiplication has the following properties*

a) $\vec{a}\,\vec{a} = |\vec{a}\,|^2$, $\vec{a}\,\vec{a} = 0$ *if and only if* $\vec{a} = \vec{0}$;

b) $(\vec{a} + \vec{b})\vec{c} = \vec{a}\vec{c} + \vec{b}\vec{c}$, $\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$,

c) $(\lambda\vec{a})\vec{b} = \lambda(\vec{a}\vec{b})$, $\vec{a}(\lambda\vec{b}) = \lambda(\vec{a}\vec{b})$.

The property a) follows from the definition of scalar product, the other properties are immediate consequences of the coordinate form of scalar product. □

For *nonzero* vectors $\vec{a} = \{a_1, a_2\}$ and $\vec{b} = \{b_1, b_2\}$ from the relation $\vec{a}\,\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$ where φ is the angle between these vectors we get that $\cos\varphi = \frac{\vec{a}\,\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$ or in the coordinate form

$$\cos\varphi = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}. \qquad (4)$$

Definition. Nonzero vectors $\vec{a}$ and $\vec{b}$ are called orthogonal if the angle between them equals $\pi/2$. The zero vector is considered orthogonal to any vector.

We will prove that the $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\,\vec{b} = 0$. This relation is valid if one of the given vectors is the zero vector. In the case $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ we denote by φ the angle between these vectors and take into account that $0 \leqslant \varphi \leqslant \pi$. We get the chain of equivalent relations:

$$\varphi = \frac{\pi}{2}, \quad \cos\varphi = 0, \quad |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi = 0, \quad \vec{a}\,\vec{b} = 0.$$

Passing to the coordinate form, we obtain the following statement: *the vectors* $\vec{a} = \{a_1, a_2\}$ *and* $\vec{b} = \{b_1, b_2\}$ *are orthogonal if and only if* $a_1b_1 + a_2b_2 = 0$.

1.2 Straight line on the plane

1.2.1 Equations of straight lines

The straight line is uniquely determined if we know some its point and a nonzero vector which is orthogonal to this line. Assume that a straight line ℓ is determined by a point M_0 and a vector $\vec{n} \neq \vec{0}$. The point M is on the line ℓ if and only if the vectors $\vec{n}$ and $\overrightarrow{M_0M}$ are orthogonal (see Fig. 13).

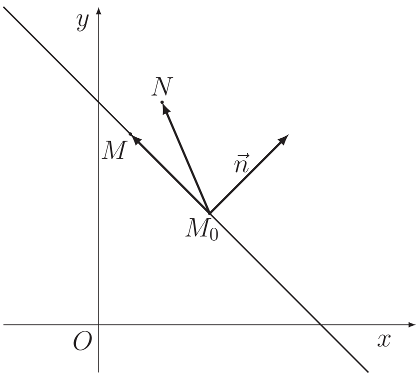

Fig. 13. Equation of the straight line:
the point M is on the line, the point N is not on the line

Therefore the equation of the line takes the vector form $\vec{n}\,\overrightarrow{M_0M} = 0$.

Assume that $M_0(x_0, y_0)$, $\vec{n} = \{A, B\}$. Then for a point $M(x, y)$ we get that $\overrightarrow{M_0M} = \{x - x_0, y - y_0\}$ and the equation of the straight line takes the form

$$A(x - x_0) + B(y - y_0) = 0, \tag{5}$$

or $Ax + By - (Ax_0 + By_0) = 0$. Denoting $C = -(Ax_0 + By_0)$ we rewrite the obtained equation in the final form $Ax + By + C = 0$.

Equations of this form are called *linear.* Since $\vec{n} \neq \vec{0}$ we have the additional condition $|A| + |B| \neq 0$ (or, equivalently $A^2 + B^2 \neq 0$). We note that such an equation of the straight line is not unique. The same line is defined by the equation $2Ax + 2By + 2C = 0$, or more generally by the equations $\alpha Ax + \alpha By + \alpha C = 0$ for arbitrary $\alpha \neq 0$[1].

The vector $\vec{n}$ defining the straight line is called its *normal vector.* It is obvious that it is defined up to a nonzero scalar factor.

We are going to give conditions when two lines are parallel, orthogonal or coincide.

Assume that we have two straight lines ℓ_1 and ℓ_2 defined by the following equations

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0.$$

Then the following assertion is valid. *The lines ℓ_1 and ℓ_2 are parallel if and only if $A_1 = \alpha A_2$, $B_1 = \alpha B_2$ for some scalar α, or equivalently $\frac{A_1}{A_2} = \frac{B_1}{B_2}$.*

The proof is reduced to the corresponding statement for the normal vectors $n_1 = \{A_1, b_1\}$, $n_2 = \{A_2, b_2\}$, since the straight lines are parallel if and only if their normal vectors are collinear. Now it remains to apply the given above collinearity condition for the case of vectors.

We apply similar arguments to obtain conditions when the given lines are orthogonal. This property is valid if and only if the normal vectors n_1 and n_2 are orthogonal.

Using the given above orthogonality condition for vectors we get that *the straight lines ℓ_1 and ℓ_2 are orthogonal if and only if $A_1A_2 + B_1B_2 = 0$.*

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[1] It will be proved below that such equations with arbitrary values of the parameter $\alpha \neq 0$ exhaust all possible linear equations of this straight line.

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