M. E. Abramyan

Lectures on differential calculus of functions of one variable

Textbook

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The textbook contains lecture material for the first semester of the course on mathematical analysis and includes the following topics: the limit of a sequence, the limit of a function, continuous functions, differentiable functions (up to Taylor's formula, L'Hospital's rule, and the study of functions by differential calculus methods). A useful feature of the book is the possibility of studying the course material at the same time as viewing a set of 22 video lectures recorded by the author and available on youtube.com. Sections and subsections of the textbook are provided with information about the lecture number, the start time of the corresponding fragment and the duration of this fragment. In the electronic version of the textbook, this information is presented as hyperlinks, allowing reader to immediately view the required fragment of the lecture.

The textbook is intended for students specializing in science and engineering.

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Contents

Preface . . . 7

Video lectures . . . 10
- Using video lectures . . . 10
- Using subtitles . . . 14

Preliminary information . . . 15
- Mathematical logic . . . 15
- Sets . . . 15
- Quantifiers . . . 16
- Absolute value and the integer part of a real number . . . 17
- Principle of mathematical induction . . . 17
- Mappings and functions . . . 18

1. Boundaries of sets . . . 20
- The continuity axiom of real numbers . . . 20
- Boundaries and exact boundaries of number sets . . . 20
- Arithmetic operations on sets . . . 24

2. Limit of a sequence . . . 27
- Neighborhood and symmetric neighborhood of a point . . . 27
- Definition of the limit of a sequence . . . 28
- The simplest properties of the limit of a sequence . . . 32

3. Properties of the limit of a sequence . . . 34
- Infinitesimal sequences: definition and properties . . . 34
- A criterion for convergence in terms of an infinitesimal . . . 35
- Arithmetic properties of the limit of a sequence . . . 35
- Passing to the limit in inequalities . . . 37

4. Infinite limits . . . 40
- Neighborhoods of the points at infinity . . . 40
- Infinitely large sequences . . . 40
- Arithmetic properties of infinitely large sequences . . . 41

5. Monotone sequences . . . 44
Bounded and monotone sequences: definitions . . . 44
Convergence of monotone sequences . . . 44
Examples of application of the convergence theorem for monotone sequences . . . 46

6. Nested segments theorem and Bolzano–Cauchy theorem on the limit point . . . 53
Nested segments theorem . . . 53
Limit points of a set. Bolzano–Cauchy theorem . . . 54

7. Subsequences. Cauchy criterion . . . 58
Subsequences. Bolzano–Weierstrass theorem . . . 58
Fundamental sequences. Cauchy criterion for sequence convergence . . . 62

8. The limit of a function . . . 66
Definition and uniqueness of the limit of a function . . . 66
Criterion for the existence of the limit of a function in terms of sequences . . . 68
Examples of functions with and without limits . . . 70
Limits at the points at infinity and infinite limits . . . 72

9. Properties of the limit of a function . . . 74
Limit of a function and arithmetic operations . . . 74
Passing to the limit of a function in inequalities . . . 75
The theorem on the limit of superposition of functions . . . 76

10. One-sided limits. Some important function limits . . . 78
Definition of one-sided limits of a function . . . 78
Criterion for the existence of the limit of a function in terms of one-sided limits . . . 79
The first remarkable limit . . . 81
The second remarkable limit . . . 83

11. The limits of monotone bounded functions. Cauchy criterion for functions . . . 87
Monotone and bounded functions . . . 87
Cauchy criterion for the existence of the function limit . . . 89

12. Continuity of function at a point . . . 92
Definition of a continuous function at a point . . . 92

Examples of continuous functions 93
Simplest properties of continuous functions 94
Arithmetic properties of continuous functions 95
Superposition of continuous functions 97

13. Continuity of a function on a set 101
Intermediate value theorem 101
Weierstrass theorems on the properties of functions continuous on a segment 104
Uniform continuity 107

14. Points of discontinuity 112
Points of discontinuity of a function, their classification and examples 112
Discontinuity points for monotone functions 114
Criterion for the continuity of a monotone function 117
Inverse function theorem 118

15. O-notation 121
Functions which are infinitesimal in comparison with other functions 121
Functions which are bounded in comparison with other functions . . . 123
Some properties related to O-notation 124
Equivalent functions at a point 125

16. Differentiable functions 128
Preliminary remarks and basic definitions 128
Continuity of a differentiable function 130
Differential of a function 131
Derivatives of some elementary functions 132

17. Properties of differentiable functions 135
Arithmetic properties of derivatives and differentials 135
Differentiation of superposition 138
Differentiation of inverse function 142

18. Hyperbolic and inverse hyperbolic functions 146
Hyperbolic functions and their properties 146
Inverse hyperbolic functions and their properties 147

19. Physical sense and geometric sense of the derivative 151
Physical sense of the derivative 151
Geometric sense of the derivative 152

20. Higher-order derivatives 155
Higher-order derivatives: definition and examples 155
Higher-order derivatives for sum and product of functions 157
Number of combinations: definition and properties 158
The Leibniz rule for the differentiation of a product 159

21. The basic theorems of differential calculus 163
Local extrema of functions. Fermat's theorem 163
Rolle's theorem, Lagrange's theorem, and Cauchy's mean value theorem 165

22. Taylor's formula 172
Taylor's formula for polynomials and for arbitrary differentiable functions 172
Various representations of the remainder term in Taylor's formula 175
Expansions of elementary functions by Taylor's formula in a neighborhood of zero 180

23. L'Hospital's rule 186
Formulation and proof of L'Hospital's rule 186
Examples of applying L'Hospital's rule 190
Supplement. An example of a differentiable function whose derivative is not continuous 191

24. Application of differential calculus to the study of functions 194
Local extrema of functions 194
Convex functions 197
Inflection points of a function 201
Location of the graph of a function relative to a tangent line 205
Asymptotes 208
Example of a function study 210

References 214

Index 216

1. Boundaries of sets

The continuity axiom of real numbers 1A/00:00 (09:55)

Real numbers have a large amount of properties associated with arithmetic operations (addition, multiplication), as well as with comparison operations. These properties are studied in detail in the course of algebra. For our purposes, the property of real numbers, called the *continuity axiom*, will play a special role.

THE CONTINUITY AXIOM OF THE SET OF REAL NUMBERS.

Let X, Y be nonempty subsets of the set $\mathbb{R}$ with the following property: for any two elements $x \in X$, $y \in Y$ the inequality $x \le y$ holds. Then there exists a number $c \in \mathbb{R}$ such that for any elements $x \in X$, $y \in Y$ the inequality $x \le c \le y$ holds.

REMARK.

The set of rational numbers does not have this property. Indeed, consider two nonempty subsets of the set of *rational* numbers:

$$X = \left\{x \in \mathbb{Q} : 1 < x < \sqrt{2}\right\}, \quad Y = \left\{y \in \mathbb{Q} : \sqrt{2} < y < 2\right\}.$$

Obviously, the inequality $x \le y$ holds for any elements $x \in X$, $y \in Y$, but there is no *rational* number c satisfying the condition $x \le c \le y$ for all $x \in X$, $y \in Y$, since the number $\sqrt{2}$ is irrational.

Boundaries and exact boundaries of number sets

Bounded sets of numbers: basic definitions 1A/09:55 (16:59)

DEFINITION.

A number set X is called *upper-bounded* (or *bounded from above*) if there exists a real number M such that for any element x from the set X the estimate $x \le M$ is true:

$$\exists\, M \in \mathbb{R} \quad \forall\, x \in X \quad x \le M.$$

If a set X is *not* bounded from above, then this means that

$$\forall\, M \in \mathbb{R} \quad \exists\, x \in X \quad x > M.$$

A number set X is called *lower-bounded* (or *bounded from below*) if there exists a real number m such that for any element x from the set X the estimate $x \geq m$ holds:

$$\exists\, m \in \mathbb{R} \quad \forall\, x \in X \quad x \geq m.$$

If the set X is *not* bounded from below, then this means that

$$\forall\, m \in \mathbb{R} \quad \exists\, x \in X \quad x < m.$$

A number set X is called *bounded* if it is upper-bounded and lower-bounded:

$$\exists\, m, M \in \mathbb{R} \quad \forall\, x \in X \quad m \leq x \leq M.$$

The number M that appears in the definition of a upper-bounded set is called the *upper bound* of this set, and the number m that appears in the definition of a lower-bounded set is called the *lower bound* of this set.

If the set X is bounded, then

$$\exists\, M_0 > 0 \quad \forall\, x \in X \quad |x| \leq M_0.$$

As M_0, one can take the maximum of the numbers $|m|$ and $|M|$, where m and M are numbers from the definition of a bounded set.

Exact boundaries of number sets: the first definition

`1A/26:54 (03:38)`

Definition 1 of supremum and infimum.

If X is an upper-bounded set, then the smallest upper bound of the set X is called the *supremum* of the set X, or its *least upper bound* (or its *exact upper bound*) and is denoted as follows: $\sup X$.

If X is a lower-bounded set, then the largest lower bound of the set X is called the *infimum* of the set X, or its *greatest lower bound* (or its *exact lower bound*) and is denoted as follows: $\inf X$.

Theorems on the existence of the exact boundaries

`1A/30:32 (06:43)`

Theorem 1 (on the existence of the least upper bound).

A nonempty set bounded from above has the least upper bound.

Proof.

Let X be a given set. Denote by B the set of all its upper bounds. The set B is not empty, since by condition X is bounded from above. Then the estimate $x \leq y$ is true for any $x \in X$, $y \in B$.

Thus, the conditions of the continuity axiom of real numbers are satisfied, if we take the set X as the set A.

By the continuity axiom, we obtain:

$$\exists\, c \in \mathbb{R} \quad \forall\, x \in X, y \in B \quad x \le c \le y.$$

So, the number c is the least upper bound, because:
1) c is the upper bound of the set X, since $\forall\, x \in X\ x \le c$,
2) c is the smallest upper bound, since $\forall\, y \in B\ c \le y$. □
The following theorem can be proved in a similar way.
Theorem 2 (on the existence of the greatest lower bound).
A nonempty set bounded from below has the greatest lower bound.
Corollary.
A nonempty bounded set X has the least upper bound and greatest lower bound.

Exact boundaries of number sets: the second definition 1B/00:00 (13:34)

Definition 2 of supremum and infimum.
The number s is called the *supremum* of a number set X if
1) this number is the upper bound:

$$\forall\, x \in X \quad x \le s;$$

2) this number is the smallest upper bound:

$$\forall\, \varepsilon > 0 \quad \exists\, x \in X \quad x > s - \varepsilon.$$

The number i is called the *infimum* of a number set X if
1) this number is the lower bound:

$$\forall\, x \in X \quad x \ge i;$$

2) this number is the largest lower bound:

$$\forall\, \varepsilon > 0 \quad \exists\, x \in X \quad x < i + \varepsilon.$$

Obviously, definitions 1 and 2 are equivalent.
Example.
Let us prove that b is the least upper bound of the interval (a, b). By the definition of an interval, the point b is the upper bound (since if $x \in (a, b)$, then $a < x < b$). It remains to show that b is the smallest upper bound, i. e. that the following condition holds:

$$\forall\, \varepsilon > 0 \quad \exists\, x \in (a, b) \quad x > b - \varepsilon.$$

Indeed, for such x it is possible, for example, to take the point $b - \frac{\varepsilon}{2}$ (or any point of the interval (a, b) if $b - \frac{\varepsilon}{2} \le a$).

Maximum and minimum elements of a set

1B/13:34 (11:02)

Definition.

Let X be a nonempty upper-bounded set. If the condition $\sup X \in X$ is fulfilled, then the element $\sup X$ is called the *maximum element* of the set X and denoted by $\max X$.

Let X be a nonempty lower-bounded set. If the condition $\inf X \in X$ is fulfilled, then the element $\inf X$ is called the *minimum element* of the set X and denoted by $\min X$.

Not each bounded nonempty set has a maximum or minimum element. For example, the interval (a, b) has neither a minimum nor a maximum element.

A set consisting of a finite number of numbers always has a minimum and maximum element. These elements can be found using the search algorithm for the minimum or maximum element.

Theorem 1 (on the existence of a maximum element in an upper-bounded integer set).

If a nonempty set X contains only integers and is bounded from above, then it has a maximum element.

Proof.

If X is upper-bounded, then it has the least upper bound s: $s = \sup X$. This means that $\forall x \in X \; x \leq s$; in addition, for $\varepsilon = 1$, there exists an element $x_0 \in X$ such that $x_0 > s - 1$.

Let us show that $x_0 = \max X$. Since $x_0 \in X$, we get: $x_0 \leq s$. The inequality $x_0 > s - 1$ can be transformed to the form $x_0 + 1 > s$, therefore all integers starting from $x_0 + 1$ do not belong to X. Thus, the estimate $x \leq x_0$ holds for all $x \in X$, which means that x_0 is the upper bound of the set X and $x_0 \geq s$. From the inequalities $x_0 \leq s$ and $x_0 \geq s$ it follows that x_0 coincides with the least upper bound s, therefore $x_0 = \max X$. □

The following theorem can be proved in a similar way.

Theorem 2 (on the existence of a minimum element in a lower-bounded integer set).

If a nonempty set X contains only integers and is bounded from below, then it has a minimum element.

Uniqueness of exact boundaries

2A/00:00 (03:23)

Theorem (on the uniqueness of exact boundaries).

If the set X has the least upper bound or the greatest lower bound, then this bound is uniquely determined.

Proof.

Let us prove this statement by contradiction. Suppose that the set X has two distinct least upper bounds: $a = \sup X$, $b = \sup X$, and $a \neq b$. Since $a \neq b$, we obtain that one of two inequalities holds: $a < b$ or $b < a$. If $a < b$ and $a = \sup X$, then the number b cannot be the least upper bound, and if $b < a$ and $b = \sup X$, then the number a cannot be the least upper bound. The obtained contradiction means that our assumption is false, and there exists the unique least upper bound.

The uniqueness of the greatest lower bound is proved similarly. □

Arithmetic operations on sets

Arithmetic operations on sets: definitions 1B/24:36 (06:09)

Definition.

Let X and Y be sets of real numbers. Then their *sum* $X + Y$ is defined as follows:

$$X + Y \stackrel{\text{def}}{=} \{z \in \mathbb{R} : (\exists x \in X, y \in Y \quad z = x + y)\}.$$

Example.

Let us find the sum of the sets $[0, 1]$ and $[2, 3]$ ($[0, 1]$ and $[2, 3]$ are segments).

For $x \in [0, 1]$, we have: $0 \le x \le 1$. For $y \in [2, 3]$, we have: $2 \le x \le 3$. Then $2 \le x + y \le 4$. Therefore, $[0, 1] + [2, 3] = [2, 4]$.

Definition.

Let X be the set of real numbers, $\lambda \in \mathbb{R}$. Then the *product* of the set X by the number λ is defined as follows:

$$\lambda X \stackrel{\text{def}}{=} \{z \in \mathbb{R} : (\exists x \in X \quad z = \lambda x)\}.$$

Remark.

Generally speaking, $X + X \neq 2X$. We give an example. Let $X = \{0, 1\}$. Then $X + X = \{0, 1, 2\}$, $2X = \{0, 2\}$. Therefore, $X + X \neq 2X$.

Theorems on the exact boundaries of the sum of sets 1B/30:45 (12:50)

Theorem 1 (on the least upper bound of the sum of sets).

Let X and Y be nonempty upper-bounded sets. Then

$$\sup(X + Y) = \sup X + \sup Y.$$

Proof.

1. Denote $s = \sup X + \sup Y$ and prove that s is an upper bound of the set $X + Y$.

We consider an arbitrary element z of the set $X + Y$: $z = x + y$ for some $x \in X$ and $y \in Y$.

Since $x \leq \sup X$, $y \leq \sup Y$, we obtain: $z = x + y \leq \sup X + \sup Y = s$.

Thus, for an arbitrary element $z \in X + Y$, the estimate $z \leq s$ holds, therefore, s is an upper bound.

2. Let us prove that s is the least upper bound of the set $X + Y$.

Let $\varepsilon > 0$ be an arbitrary positive number.

We will show that $s - \varepsilon$ is not the upper bound of the set $X + Y$, that is, there exists a number $z_0 = x_0 + y_0 \in X + Y$ such that $z_0 > s - \varepsilon$.

By the definition of the least upper bound of the set X, we have:

$$\exists\, x_0 \in X \quad x_0 > \sup X - \frac{\varepsilon}{2}.$$

By the definition of the least upper bound of the set Y, we have:

$$\exists\, y_0 \in Y \quad y_0 > \sup Y - \frac{\varepsilon}{2}.$$

Summing up these inequalities term by term, we obtain the required result:

$$z_0 = x_0 + y_0 > \sup X + \sup Y - \varepsilon = s - \varepsilon. \ \square$$

The following theorem can be proved in a similar way.

Theorem 2 (on the greatest lower bound of the sum of sets).

Let X and Y be nonempty lower-bounded sets. Then

$$\inf(X + Y) = \inf X + \inf Y.$$

Theorems on the exact boundaries of the product of a set by a number

2A/03:23 (04:13)

Theorem 1 (first theorem on the exact boundaries of the product of a set by a number).

Let X be a nonempty upper-bounded set, $\lambda > 0$. Then

$$\sup(\lambda X) = \lambda \sup X.$$

Proof.

1. Let $\lambda x \in \lambda X$.

Since for $x \in X$ we have $x \leq \sup X$, we obtain: $\lambda x \leq \lambda \sup X$.

Therefore, $\lambda \sup X$ is an upper bound of the set λX.

2. Let us choose $\varepsilon > 0$.

By the definition of the least upper bound of the set X, we have

$$\exists\, x' \in X \quad x' > \sup X - \frac{\varepsilon}{\lambda}.$$

Consequently,

$$\lambda x' > \lambda \left(\sup X - \frac{\varepsilon}{\lambda} \right) = \lambda \sup X - \varepsilon.$$

Thus, we found the element $\lambda x' \in \lambda X$ such that the inequality $\lambda x' > \lambda \sup X - \varepsilon$ holds for the selected ε. Therefore, $\lambda \sup X$ is the least upper bound of the set λX. □

The following theorem can be proved in a similar way.

Theorem 2 (second theorem on the exact boundaries of the product of a set by a number).

1. Let X be a nonempty lower-bounded set, $\lambda > 0$. Then

$$\inf(\lambda X) = \lambda \inf X.$$

2. Let X be a nonempty upper-bounded set, $\lambda < 0$. Then

$$\inf(\lambda X) = \lambda \sup X.$$

3. Let X be a nonempty lower-bounded set, $\lambda < 0$. Then

$$\sup(\lambda X) = \lambda \inf X.$$

2. Limit of a sequence

Neighborhood and symmetric neighborhood of a point

Neighborhood and symmetric neighborhood: definition and properties

2A/07:36 (13:32)

Definition.

Let A be a point on a number line: $A \in \mathbb{R}$. The *neighborhood* U_A of the point A is any interval (a, b) containing this point. The *symmetric* ε*-neighborhood* U_A^ε of the point A is the interval $(A - \varepsilon, A + \varepsilon)$, where $\varepsilon > 0$ is a number called the *radius* of the symmetric neighborhood.

The intersection of any nonempty finite set of neighborhoods of the point A is a neighborhood of the point A. The intersection of any nonempty finite set of symmetric neighborhoods of the point A is a symmetric neighborhood of the point A.

The union of any nonempty (not necessarily finite) set of neighborhoods of A is a neighborhood of A. The union of any nonempty (not necessarily finite) set of symmetric neighborhoods of A is a symmetric neighborhood of A.

Remark.

Any neighborhood (a, b) of the point A contains a symmetric neighborhood:

$$(a, b) \supset (A - \varepsilon, A + \varepsilon), \text{ where } \varepsilon = \min\{|A - a|, |A - b|\}.$$

Supplement. Intersection of neighborhoods

3A/00:00 (01:21)

In describing the properties of neighborhoods of points, we noted that the intersection of any nonempty finite set of neighborhoods of a given point is a neighborhood of this point. Now we show that in the case of an infinite set of neighborhoods, this statement is not true. To do this, it's enough to give an example.

Consider the set of intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$. All such intervals are neighborhoods of the point 0. However, their intersection consists of a single point 0. Indeed, for any point $x \neq 0$, there exists a number $n_0 \in \mathbb{N}$ such that $|x| \geq \frac{1}{n_0}$. So, the point x does not belong to the interval $\left(-\frac{1}{n_0}, \frac{1}{n_0}\right)$, and therefore it does not belong to the intersection of all such intervals for n from 1 to ∞.

Thus, the intersection of all intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, consists of a single point 0. But a single point is not a neighborhood. So, we have shown that the intersection of an infinite number of neighborhoods of a point will not necessarily be its neighborhood.

Definition of the limit of a sequence

Sequence: definition and examples

2A/21:08 (06:29)

DEFINITION.

The map $f : \mathbb{N} \to X$, where $\mathbb{N}$ is the set of natural numbers, is called the *sequence of elements* (or *terms*) $x_1 = f(1)$, $x_2 = f(2)$, ..., $x_n = f(n)$, ... and denoted by $\{x_n\}$. An element x_n is called the *common term* of the sequence.

A sequence is called a *numerical* one if $X = \mathbb{R}$.

EXAMPLES OF SEQUENCES.

$\left\{\frac{1}{n}\right\} : 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

$\left\{n^2\right\} : 1, 4, 9, 16, \ldots, n^2, \ldots$

How to define the limit of a sequence?

2A/27:37 (07:52)

If we consider the sequence $\left\{\frac{1}{n}\right\}$ and go through its elements in ascending order of their indices, then they will come closer and closer to the point 0. It is natural to assume that the number 0 will be the *limit* of the sequence $\left\{\frac{1}{n}\right\}$.

Another example of a sequence whose limit is 0 is the sequence $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$. This sequence is interesting in that its elements approach the point 0 from different sides.

If we consider the sequence $\left\{n^2\right\}$, then its elements will not approach any finite number, so it is natural to assume that this sequence has no finite limit.

What property of point 0 allows us to consider it as the limit of the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{(-1)^n}{n}\right\}$? To describe such a property, it is easiest to use the notion of a neighborhood of a point. The point A will be the limit of the sequence $\{x_n\}$ if for any neighborhood U_A of this point all elements of the sequence, except, perhaps, a finite number of its initial elements, will lie in this neighborhood. In other words, it is required that any neighborhood U_A contains an infinite number of elements of the sequence $\{x_n\}$, and outside it there is a finite number of elements.

It is easy to see that only the point 0 satisfies the indicated condition for the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{(-1)^n}{n}\right\}$.

In this definition, it is important not only that in any neighborhood there is an infinite number of elements of the sequence, but also that only a finite number remains outside the neighborhood. Without the second condition, it would turn out that the sequence $\{(-1)^n\} = \{1, -1, 1, -1, \dots\}$ has two limits: -1 and 1, however, the presence of several limits of one sequence would lead to problems in constructing the theory of limits.

Symmetric neighborhoods can also be used in the definition of the limit; this version of definition is often more convenient to use.

Definition of the limit of a sequence in the language of neighborhoods

2A/35:29 (05:33), 2B/00:00 (01:07)

Definition 1 of the sequence limit (in the language of neighborhoods).

The number $A \in \mathbb{R}$ is called the *limit* of a sequence $\{x_n\}$ if for any neighborhood U_A of the point A there exists a natural number $N \in \mathbb{N}$ such that all elements x_n with numbers greater than N will be contained in the neighborhood U_A. Formally we may write the previous condition as follows:

$$\forall\, U_A \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in U_A.$$

Definition of the limit of a sequence in the language of symmetric neighborhoods

2B/01:07 (19:41)

Definition 2 of the sequence limit (in the language of symmetric neighborhoods).

The number $A \in \mathbb{R}$ is called the *limit* of a sequence $\{x_n\}$ if for any ε-neighborhood V_A^ε of the point A with radius $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that all elements x_n with numbers greater than N will be contained in the neighborhood V_A^ε:

$$\forall\, V_A^\varepsilon \quad \exists\, N \in \mathbb{N} \quad \forall\, n > N \quad x_n \in V_A^\varepsilon.$$

Theorem (on the equivalence of two definitions of the limit of a sequence).

Definitions 1 and 2 of the limit of a sequence are equivalent.

Proof.

Obviously, if A is the limit of a sequence in the sense of definition 1, then A is also the limit in the sense of definition 2, since any symmetric neighborhood is a neighborhood.

Let us prove the opposite. Let A be the limit of $\{x_n\}$ in the sense of definition 2. We show that A is the limit of $\{x_n\}$ in the sense of definition 1.

Let U_A be an arbitrary neighborhood of A. We can choose the symmetric neighborhood V_A^ε containing in U_A: $V_A^\varepsilon \subset U_A$.

According to definition 2, for the neighborhood of V_A^ε there exists $N \in \mathbb{N}$ such that $x_n \in V_A^\varepsilon$ for all $n > N$. But $V_A^\varepsilon \subset U_A$, so $x_n \in U_A$ for all $n > N$. Thus, since the choice of the neighborhood U_A is arbitrary, the point A is also the limit in the sense of definition 1. □

Definition 2 can be reformulated as follows.

Definition 3 of the sequence limit (in the language ε–N).

The number $A \in \mathbb{R}$ is called the *limit* of a sequence $\{x_n\}$ if for any number $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that for any $n > N$ the following inequality holds: $A-\varepsilon < x_n < A+\varepsilon$, or, equivalently, $|x_n - A| < \varepsilon$:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |x_n - A| < \varepsilon.$$

Such a definition is called a definition in the language ε–N.

Limit notations: $\lim_{n\to\infty} x_n = A$, $\lim\limits_{n\to\infty} x_n = A$ or $x_n \to A$ as $n \to \infty$ ("x_n approaches A as n approaches infinity").

A sequence with a limit $A \in \mathbb{R}$ is called a *convergent* one (to the limit A).

Examples of finding the limit of the sequence using the definition

2B/20:48 (11:47)

1. $x_n = \frac{1}{n}$.

We will show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Let us select an arbitrary $\varepsilon > 0$ and find N such that for all $n > N$ the estimate $\left|\frac{1}{n} - 0\right| < \varepsilon$ holds, that is, $\frac{1}{n} < \varepsilon$.

The inequality $\frac{1}{n} < \varepsilon$ is equivalent to the inequality $n > \frac{1}{\varepsilon}$.

Let $N = \left[\frac{1}{\varepsilon}\right]$, where $[x]$ is the integer part of the number x.

Taking into account that n is natural, we get that for all $n > \left[\frac{1}{\varepsilon}\right]$ the following estimate holds: $n \ge \left[\frac{1}{\varepsilon}\right] + 1$.

This estimate can be continued if we use the property of the integer part of a real number ($[x] \le x < [x] + 1$):

$$n \ge \left[\frac{1}{\varepsilon}\right] + 1 > \frac{1}{\varepsilon}.$$

We have obtained that for all natural numbers $n > N$, where $N = \left[\frac{1}{\varepsilon}\right]$, the estimate $n > \frac{1}{\varepsilon}$ holds.

Therefore,

$$\forall\,\varepsilon > 0 \quad \exists\,N = \left[\frac{1}{\varepsilon}\right] \quad \forall\,n > N \quad \frac{1}{n} < \varepsilon.$$

This means that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. $x_n = \frac{(-1)^n}{n}$.

In this case, the limit will also be 0.

The proof is completely similar to the proof the sequence from the example 1, since the inequality $\left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$ may be written in the same form as in the example 1: $\frac{1}{n} < \varepsilon$.

Example of a sequence without limit

2B/32:35 (08:29)

We can say that the number A is the limit of a sequence $\{x_n\}$ if *any* neighborhood of the number A contains *all* elements of the sequence except, perhaps, some *finite* amount of its starting elements.

In order to show that the number A is *not* the limit of a sequence $\{x_n\}$, it suffices to select *some* neighborhood of the number A, outside which there is an *infinite* number of elements of the sequence $\{x_n\}$.

Formally, the statement that the number A is *not* the limit of a sequence $\{x_n\}$ can be written by applying the negation operation to one of definitions of the limit, for example (for definition 3):

$$\overline{\forall\,\varepsilon > 0 \quad \exists\,N \in \mathbb{N} \quad \forall\,n > N \quad |x_n - A| < \varepsilon,}$$

$$\exists\,\varepsilon > 0 \quad \forall\,N \in \mathbb{N} \quad \exists\,n > N \quad |x_n - A| \ge \varepsilon.$$

Let $\varphi_n = (-1)^n : -1, 1, -1, 1, \ldots$

Let us prove that this sequence has no limit. To do this, we use the above negation of the statement that the number A is the limit of the sequence $\{\varphi_n\}$.

Let $A = 1$. Choose $\varepsilon = \frac{1}{2}$. Then for *any* natural number N there exists an *odd* number $n > N$, for which $\varphi_n = -1$ and, therefore, this element of the sequence is *not* contained in the ε-neighborhood of the point 1. Therefore, the number $A = 1$ is not the limit of the sequence $\{\varphi_n\}$.

Let $A = -1$. Then, choosing $\varepsilon = \frac{1}{2}$, we obtain that for *any* natural number N there exists an *even* number $n > N$, for which $\varphi_n = 1$ and, therefore, this element of the sequence is *not* contained in the ε-neighborhood of the point -1. Therefore, the number $A = -1$ is also not the limit of the sequence $\{\varphi_n\}$.

Let A be a number other than 1 and -1. Let $\varepsilon = \min\{|A-1|, |A+1|\}$. Then for the ε-neighborhood of the point A, *all* elements of the sequence $\{\varphi_n\}$ will be out of this neighborhood. Therefore, all such numbers also cannot be the limit of the sequence $\{\varphi_n\}$.

The simplest properties of the limit of a sequence

The uniqueness theorem for the limit of a convergent sequence

3A/01:21 (13:39)

Theorem (on the uniqueness of the limit of a convergent sequence).

A convergent sequence cannot have two different limits.

Proof.

We prove the theorem by contradiction. Suppose that A and B are different limits of the given sequence $\{x_n\}$:

$$\lim_{n\to\infty} x_n = A, \quad \lim_{n\to\infty} x_n = B, \quad A \neq B.$$

Then the points A and B have disjoint neighborhoods U_A and U_B: $U_A \cap U_B = \varnothing$.

By the definition of the limit of a sequence, we have for the neighborhood U_A:

$$\exists\, N_1 \in \mathbb{N} \quad \forall\, n > N_1 \quad x_n \in U_A. \tag{1}$$

Similarly, for the neighborhood U_B, we have:

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n > N_2 \quad x_n \in U_B. \tag{2}$$

Let $N = \max\{N_1, N_2\}$. Then, by virtue of relations (1) and (2), $x_n \in U_A \cap U_B$ for $n > N$.

But the neighborhoods of U_A and U_B do not intersect. That means that for $n > N$ $x_n \in \varnothing$, which is impossible. The obtained contradiction means that our assumption was incorrect, and the sequence $\{x_n\}$ cannot have two different limits. □

A theorem on the boundedness of a convergent sequence

3A/15:00 (12:09)

Definition.

A sequence $\{x_n\}$ is called *bounded* if there exists $M > 0$ such that for all $n \in \mathbb{N}$ the estimate $|x_n| \leq M$ holds:

$$\exists\, M > 0 \quad \forall\, n \in \mathbb{N} \quad |x_n| \le M.$$

THEOREM (ON THE BOUNDEDNESS OF A CONVERGENT SEQUENCE).
A convergent sequence is bounded.

PROOF.

Let $A = \lim_{n\to\infty} x_n$. Then for $\varepsilon = 1$ we have:

$$\exists\, N \in \mathbb{N} \quad \forall\, n > N \quad |x_n - A| < 1.$$

Applying the triangle inequality for the absolute value of sum, we get:

$$|x_n| = |(x_n - A) + A| \le |x_n - A| + |A| < 1 + |A|.$$

Thus, for any $n > N$ we have $|x_n| < M_1$, where $M_1 = 1 + |A|$.

In addition, the set $\{|x_1|, |x_2|, \ldots, |x_N|\}$ is finite and therefore has the maximum element with the value M_2. So, the estimate $|x_n| \le M_2$ holds for all $n \le N$.

Taking $M = \max\{M_1, M_2\}$, we get:

$$\forall\, n \in \mathbb{N} \quad |x_n| \le M. \ \square$$

REMARK.

The converse assertion is not true: the bounded sequence is not necessarily convergent. As an example, we can use the previously considered sequence $\{\varphi_n\} = \{(-1)^n\}$. Obviously, it is bounded, since $\forall\, n \in \mathbb{N}$ $|\varphi_n| \le 1$, but we have proved that it has no limit.