Wavelets in Electromagnetics and Device Modeling

GEORGE W. PAN
Arizona State University
Tempe, Arizona

WILEY-INTERSCIENCE
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Dedicated to my father Pan Zhen and mother Lei Tian-Lu

Contents

Preface

Applied mathematics has made considerable progress in wavelets. In recent years interest in wavelets has grown at a steady rate, and applications of wavelets are expanding rapidly. A virtual flood of engineers, with little mathematical sophistication, is about to enter the field of wavelets. Although more than 100 books on wavelets have been published since 1992, there is still a large gap between the mathematician's rigor and the engineer's interest. The present book is intended to bridge this gap between mathematical theory and engineering applications.

In an attempt to exploit the advantages of wavelets, the book covers basic wavelet principles from an engineer's point of view. With a minimum number of theorems and proofs, the book focuses on providing physical insight rather than rigorous mathematical presentations. As a result the subject matter is developed and presented in a more basic and familiar way for engineers with a background in electromagnetics, including linear algebra, Fourier analysis, sampling function of $\sin \pi x/\pi x$, Dirac δ function, Green's functions, and so on. The multiresolution analysis (MRA) is naturally delivered in Chapter 2 as a basic introduction that shows a signal decomposed into several resolution levels. Each level can be processed according to the requirement of the application. The application of MRA lies within the Mallat decomposition and reconstruction algorithm. MRA is further explained in a fast wavelet transform section with an example of frequency-dependent transmission lines. Mathematically elegant proofs and derivations are presented in a smaller font if their content is beyond the engineering requirement. Readers with no time or interest in this depth of mathematics may always skip the paragraphs or sections written in smaller font without jeopardizing their understanding of the main subjects.

The main body of the book came from conference presentations, including the IEEE Microwave Theory and Techniques Symposium (IEEE-MTT), IEEE Antennas and Propagation (IEEE-AP), Radio Science (URSI), IEEE Magnetics, Progress in Electromagnetic Research Symposium (PIERS), Electromagnetic and Light Scattering (ELS), COMPUMAG, Conference on Electromagnetic Field Computation (CEFC), Association for Computational Electromagnetic Society (ACES), International Conference on Microwave and Millimeter Wave Technology (ICMTT), and

International Conference on Computational Electromagnetics and its Applications (ICCEA). The book has evolved from curricula taught at the graduate level in the Department of Electronic Engineering at Canterbury University (Christchurch, New Zealand) and Arizona State University. The material was taught as short courses at Moscow State University, CSIRO (Sydney, Australia), IEEE Microwave Theory and Techniques Symposium, Beijing University, Aerospace 207 Institute, and the 3rd Institute of China. The participants in these courses were electrical engineering and computer science students as well as practicing engineers in industry. These people had little or no prior knowledge of wavelets.

The book may serve as a reference book for engineers, practicing scientists, and other professionals. Real-world state-of-the-art issues are extensively discussed, including full-wave modeling of coupled lossy and dispersive transmission lines, scattering of electromagnetic waves from 2D/3D bodies and from randomly rough surfaces, radiation from linear and patch antennas, and modeling of 2D semiconductor devices. The book can also be used as a textbook, as it contains questions, working examples, and 11 exercise assignments with a solution manual. It has been used several times in teaching a one-semester graduate course in electrical engineering.

The book consists of 10 chapters. The first six chapters are dedicated to basic theory and training, followed by four chapters in real-world applications. Chapter 1 summarizes mathematical preliminaries, which may be skipped on the first reading. Chapter 2 provides some background and theoretical insights. Chapter 3 covers the basic orthogonal wavelet theory. Other wavelet topics are discussed in Chapters 4 through 10, including biorthogonal wavelets, weighted wavelets, interpolating wavelets, Green's wavelets, and multiwavelets. Chapter 4 presents applications of wavelets in solving integral equations. Special treatments of edges are discussed here, including periodic wavelets and intervallic wavelets. Chapter 5 derives the positive sampling functions and their biorthogonal counterparts using Daubechies wavelets. Many advantages derive from the use of the sampling biorthogonal time domain (SBTD) method to replace the finite difference time domain (FDTD) scheme. Chapter 6 studies multiwavelet theory, including biorthogonal and orthogonal multiwavelets with applications in the edge-based finite element method (EEM). Advanced topics are presented in Chapter 7, 8, and 9, respectively, for scattering and radiation, 3D rough surface scattering, packaging and interconnects. Chapter 10 is devoted to semiconductor device modeling using the aforementioned knowledge of wavelets. Numerical procedures are fully detailed so as to help interested readers develop their own algorithms and computer codes.

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GEORGE W. PAN

Tempe, Arizona

CHAPTER ONE

Notations and Mathematical Preliminaries

1.1 NOTATIONS AND ABBREVIATIONS

The notations and abbreviations used in the book are summarized here for ease of reference.

$D^{(\alpha)} f = f^{\alpha}(t) := df^{\alpha}(t)/dt^{\alpha}$
$\bar{f}$—complex conjugate of f
$\hat{f} := \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$, Fourier transform of $f(t)$
$f(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\, d\omega$, inverse Fourier transform of $\hat{f}(\omega)$
$\|f\|$—norm of a function
$f * g$—convolution
$\langle f, h\rangle := \int \overline{f(t)}h(t)\, dt$, inner product
$f_n = O(n)$-order of n, $\exists C$ such that $f_n \leq Cn$
$\mathcal{C}$—complex
N—nonnegative integers
R—real number
R^n—real numbers of size n
Z—integers
Z^+—positive integers
$L^2(R)$—functional space consisting finite energy functions $\int |f(t)|^2\, dt < +\infty$
$L^p(R)$—function space that $\int |f(t)|^p\, dt < +\infty$
$l^2(Z)$—finite energy series $\sum_{n=-\infty}^{\infty} |a_n|^2 < +\infty$
Ω—set
$H^s(\Omega) := W^{s,2}(\Omega)$-Sobolev space equipped with inner product of
$\langle u, v\rangle_{s,2} := \sum_{|\alpha|\leq s} \int_{\Omega} \overline{D^{\alpha}u} D^{\alpha} v d\Omega$

$V \oplus W$—direct sum
$V \otimes W$—tensor product
∇f—gradient
$\vec{H}, \vec{E}$—vector fields
$\nabla \times \vec{H}$—curl
$\nabla \cdot \vec{E}$—divergence
$\lfloor \alpha \rfloor$—largest integer $m \leq \alpha$
$\delta_{m,n}$—Kronecker delta
$\delta(t)$—Dirac delta
$\chi[a, b]$—characteristic function, which is 1 in $[a, b]$ and zero outside
$\square$—end of proof
$\exists$—exist
$\forall$—any
iff—if and only if
a.e.—almost everywhere
d.c.—direct current
o.n.—orthonormal
o.w.—otherwise

1.2 MATHEMATICAL PRELIMINARIES

This chapter is arranged here to familiarize the reader with the mathematical notation, definitions and theorems that are used in wavelet literature and in this book. Important mathematical concepts are briefly reviewed. In most cases no proof is given. For more detailed discussions or in depth studies, readers are referred to the corresponding references [1–5].

Readers are suggested to skip this chapter in their first reading. They may then return to the relevant sections of this chapter if unfamiliar mathematical concepts present themselves during the course of the book.

1.2.1 Functions and Integration

A function $f(t)$ is called integrable if

$$\int_{-\infty}^{\infty} |f(t)|\, dt < +\infty, \tag{1.2.1}$$

and we say that $f \in L^1(R)$.

Two functions $f_1(t)$ and $f_2(t)$ are equal in $L^1(R)$ if

$$\int_{-\infty}^{\infty} |f_1(t) - f_2(t)|\, dt = 0.$$

This implies that $f_1(t)$ and $f_2(t)$ may differ only on a set of points of zero measure. The two functions f_1 and f_2 are almost everywhere (a.e.) equal.

Fatou Lemma. Let $\{f_n\}_{n\in N}$ be a set of positive functions. If

$$\lim_{n\to\infty} f_n(t) = f(t)$$

almost everywhere, then

$$\int_{-\infty}^{\infty} f(t)\,dt \le \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\,dt.$$

This lemma provides an inequality when taking a limit under the Lebesgue integral for positive functions.

Lebesgue Dominated Convergence Theorem. Let $f_k(t) \in L(E)$ for $k = 1, 2, \ldots,$ and

$$\lim_{k\to\infty} f_k(t) = f(t) \quad \text{a.e.}$$

If there exists an integrable function $F(t)$ such that

$$|f_k(t)| \le F(t) \quad \text{a.e.}, \quad k = 1, 2, \ldots,$$

then

$$\lim_{k\to\infty} \int_E f_k(t)\,dt = \int_E f(t)\,dt.$$

This theorem allows us to exchange the limit with integration.

Fubini Theorem. If

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t_1, t_2)\,dt_1 \right) dt_2 < \infty,$$

then

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)\,dt_1\,dt_2 &= \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} f(t_1, t_2)\,dt_1 \\ &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} f(t_1, t_2)\,dt_2. \end{aligned}$$

This theorem provides a sufficient condition for commuting the order of the multiple integration.

1.2.2 The Fourier Transform

The Fourier transform pair is defined as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt,$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega.$$

Rigorously speaking, the Fourier transform of $f(t)$ exists if the Dirichlet conditions are satisfied, that is,

(1) $\int_{-\infty}^{\infty} |f(t)|\, dt < +\infty$, as in (1.2.1).
(2) $f(t)$ has a finite number of maxima and minima within any finite interval, and any discontinuities of $f(t)$ are finite. There are only a finite number of such discontinuities in any finite interval.

All functions satisfying (1.2.1) form a functional space L^1. A weaker condition for the existence of the Fourier transform of $f(t)$, in replace of (1.2.1), is given as

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt < +\infty. \tag{1.2.2}$$

All functions satisfying (1.2.2) form a functional space L^2.

When the Dirichlet conditions are satisfied, the inverse Fourier transform converges to $f(t)$ if $f(t)$ is continuous at t, or to

$$\frac{f(t^+) + f(t^-)}{2}$$

if $f(t)$is discontinuous at t. When $f(t)$ has infinite energy, its Fourier transform may be defined by incorporating generalized functions. The resultant is called the generalized Fourier transform of the original function.

1.2.3 Regularity

Lipschitz Regularity. If a function $f(t)$ has a singularity at $t = v$, this implies that $f(t)$ is not differentiable at v. Lipschitz exponent at v characterizes the singularity behavior.

The Taylor expansion relates the differentiability of a function to a local polynomial approximation. Suppose that f is m times differentiable in $[v - h, v + h]$. Let p_v be the Taylor polynomial in the neighborhood of v:

$$p_v(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(v)}{k!} (t - v)^k.$$

Then the error

$$|\varepsilon_v(t)| \leq \frac{|t-v|^m}{m!} \sup_{u\in[v-h,v+h]} |f^{(m)}(u)|$$

where

$$t \in [v-h, v+h], \quad \varepsilon_v(t) := f(t) - p_v(t).$$

The Lipschitz regularity refines the upper bound on the error $\varepsilon_v(t)$ with noninteger exponents. Lipschitz exponents are also referred to as Hölder exponents.

Definition 1 (Lipschitz). A function $f(t)$ is pointwise Lipschitz $\alpha \geq 0$ at $t = v$, if there exist $M > 0$ and a polynomial $p_v(t)$ of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in R, \quad |f(t) - p_v(t)| \leq M|t-v|^\alpha. \tag{1.2.3}$$

Definition 2. A function $f(t)$ is uniformly Lipschitz α over $[a, b]$ if it satisfies (1.2.3) for all $v \in [a, b]$ with a constant M independent of v.

Definition 3. The Lipschitz regularity of $f(t)$ at v or over $[a, b]$ is the *sup* of the α such that $f(t)$ is Lipschitz α.

Theorem 1. A function $f(t)$ is bounded and uniform Lipschitz α over R if

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|(1 + |\omega|^\alpha)\, d\omega < +\infty. \tag{1.2.4}$$

If $0 \leq \alpha < 1$, then $p_v(t) = f(v)$ and the Lipschitz condition reduces to

$$\forall t \in R, \quad |f(t) - f(v)| \leq M|t-v|^\alpha.$$

Here the function is bounded but discontinuous at v, and we say that the function is Lipschitz 0 at v.

Proof. When $0 \leq \alpha < 1$, it follows $m := \lfloor \alpha \rfloor = 0$, and $p_v(t) = f(v)$.
The uniform Lipschitz regularity implies that $\exists M > 0$ such that

$$\forall (t, v) \in R^2.$$

We need to have

$$\frac{|f(t) - f(v)|}{|t-v|^\alpha} \leq M.$$

Since

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega,$$

$$\frac{|f(t)-f(v)|}{|t-v|^\alpha} = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\hat{f}(\omega)\left[\frac{e^{i\omega t}}{|t-v|^\alpha} - \frac{e^{i\omega v}}{|t-v|^\alpha}\right]d\omega\right|$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|\frac{|e^{i\omega t}-e^{i\omega v}|}{|t-v|^\alpha}\,d\omega.$$

(1) For $|t-v|^{-1} \leq |\omega|$,

$$\frac{|e^{i\omega t}-e^{i\omega v}|}{|t-v|^\alpha} \leq \frac{2}{|t-v|^\alpha} \leq 2|\omega|^\alpha.$$

(2) For $|t-v|^{-1} \geq |\omega|$,

$$|e^{i\omega t}-e^{i\omega v}| = \left|i\omega(t-v) - \frac{\omega^2}{2!}(t-v)^2 - i\frac{(t-v)^3}{3!} + - \cdots\right|.$$

On the right-hand side of the equation above, the imaginary part

$$I = \omega(t-v) - \frac{[\omega(t-v)]^3}{3!} + \frac{[\omega(t-v)]^5}{5!} - + \cdots \leq \omega(t-v),$$

and the magnitude of the real part

$$R = \left\{\frac{[\omega(t-v)]^2}{2!} - \frac{[\omega(t-v)]^4}{4!} + - \cdots\right\} \leq \frac{[\omega(t-v)]^2}{2!}.$$

Thus

$$|(t-v)\omega| \leq 1 \quad \text{and} \quad [(t-v)\omega]^2 \leq |(t-v)\omega|$$

and

$$\begin{aligned}|e^{i\omega t}-e^{i\omega v}| &\leq \left|i\omega(t-v) + \frac{[\omega(t-v)]^2}{2!}\right|\\ &= \sqrt{[\omega(t-v)]^2 + \frac{\omega^4(t-v)^4}{4}}\\ &\leq |2\omega(t-v)|.\end{aligned}$$

Hence

$$\frac{|e^{i\omega t}-e^{i\omega v}|}{|t-v|^\alpha} \leq \frac{2|\omega||t-v|}{|t-v|^\alpha} \leq 2|\omega|^\alpha.$$

Combining (1) and (2) yields

$$\frac{|f(t) - f(v)|^2}{|t - v|^\alpha} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} 2|\hat{f}(\omega)|\ |\omega|^\alpha\, d\omega := M.$$

It can be verified that if

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|[1 + |\omega|^p]\, d\omega < \infty,$$

then $f(t)$ is p times continuously differentiable. Therefore, if

$$\int_{-\infty}^{\infty} \hat{f}(\omega)[1 + |\omega|^\alpha]\, d\omega < \infty,$$

then $f^{(m)}(t)$ is uniformly Lipschitz $\alpha - m$, and hence $f(t)$ is uniformly Lipschitz α, where $m = \lfloor \alpha \rfloor$. □

1.2.4 Linear Spaces

Linear Space. A linear space H is a nonempty set. Let $\mathcal{C}$ be complex. H is called a complex linear space if

(1) $x + y = y + x$.
(2) $(x + y) + z = x + (y + z)$.
(3) There exists a unique element $\theta \in H$ such that for $\forall x \in H, x + \theta = \theta + x$.
(4) For $\forall x \in H$, there exists a unique $-x$ such that $x + (-x) = \theta$.

In addition we define scalar multiplication $\forall(\alpha, x) \in \mathcal{C} \times H$ such that

(1) $\alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in \mathcal{C}, \forall x \in H$.
(2) $1x = x$.
(3) $(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in \mathcal{C}, \forall x \in H$.
$\alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in \mathcal{C}, \forall x, y \in H$.

Norm of a Vector

Definition. Mapping of $\| x \|: R^n \to R$ is called the norm of x on R^n iff

(1) $\| x \| \geq 0, \forall x \in R^n$.
(2) $\| \alpha x \| = |\alpha|\ \| x \|, \forall \alpha \in R, x \in R^n$.
(3) $\| x + y \| \leq \| x \| + \| y \|, \forall x, y \in R^n$.
(4) $\| x \| = 0 \Longleftrightarrow x = 0$.

Let $x = (x_1, x_2, \ldots, x_n)^T \in R^n$. The following are commonly used norms:

$$\| x \|_\infty = \max_i |x_i|, \qquad \ell^\infty \text{ norm},$$

$$\| x \|_1 = \sum_{i=1}^{n} |x_i|, \qquad \ell^1 \text{ norm},$$

$$\| x \|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}, \qquad \ell^2 \text{ norm},$$

$$\| x \|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \qquad \ell^p \text{ norm}.$$

1.2.5 Functional Spaces

Metric, Banach, Hilbert, and Sobolev spaces are functional spaces. A functional space is a collection of functions that possess a certain mathematical structure pattern.

Metric Space. A metric space H is a nonempty set that defines the distance of a real-valued function $\rho(x, y)$ that satisfies:

(1) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ iff $x = y$.
(2) $\rho(x, y) = \rho(y, x)$.
(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y), \forall x, y, z \in H$.

Banach Space. Banach space is a vector space H that admits a norm, $\| \cdot \|$, that satisfies:

(1) $\forall f \in H, \| f \| \geq 0$ and $\| f \| = 0$ iff $f = 0$.
(2) $\forall \alpha \in \mathcal{C}, \| \alpha f \| = |\alpha| \, \| f \|$.
(3) $\| f + g \| \leq \| f \| + \| g \|, \forall f, g \in H$.

These properties of norms are similar to those of distance, except the homogeneity of (2) is not required in defining a distance. The convergence of $\{f_n\}_{n \in N}$ to $f \in H$ implies that $\lim_{n \to \infty} \| f_n - f \| = 0$ and is denoted as $\lim_{n \to \infty} f_n = f$.

To guarantee that we remain in H when taking the limits, we define the Cauchy sequences. A sequence $\{f_n\}_{n \in N}$ is a Cauchy sequence if for $\forall \varepsilon > 0$, there exist n and m large enough such that $\| f_m - f_n \| < \varepsilon$. The space H is said to be complete if every Cauchy sequence in H converges to an element of H. A complete linear space equipped with norm is called the Banach space.

Example 1 Let S be a collection of sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$. We define addition and multiplication naturally as

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, \dots),$$
$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n, \dots),$$

and define distance as

$$\rho(x, y) = \sum \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

It can be verified that such a space S is not a Banach space, because $\rho(x, y)$ does not satisfy the homogeneous condition of the norm.

Example 2 For any integer p we define over discrete sequence f_n the norm

$$\|f\|_p = \left[\sum_{n=-\infty}^{\infty} |f_n|^p \right]^{1/p}.$$

The space $\ell^p = \{f : \|f\|_p < \infty\}$ is a Banach space with norm $\|f\|_p$.

Example 3 The space $L^p(R)$ is composed of measurable functions f on R that

$$\| f \|_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p \right\}^{1/p} < \infty.$$

The space $L^p(R) = \{f : \| f \|_p < \infty\}$ is a Banach space.

Hilbert Space. A Hilbert space is an inner product space that is complete. The inner product satisfies:

(1) $\langle \alpha f + \beta g, h \rangle = \overline{\alpha} \langle f, g \rangle + \overline{\beta} \langle g, h \rangle$ for $\alpha, \beta, \in \mathcal{C}$ and $f, g, h \in H$.
(2) $\langle f, g \rangle = \overline{\langle g, f \rangle}$.
(3) $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ iff $f = 0$. One may verify that

$$\| f \| = \langle f, f \rangle^{1/2}$$

is a norm.
(4) The Cauchy–Schwarz inequality states that

$$|\langle f, g \rangle| \leq \| f \| \| g \|,$$

where the equality is held iff f and g are linearly dependent.

In a Banach space the norm is defined, which allows us to discuss the convergence. However, the angles and orthogonality are lacking. A Hilbert space is a Banach space equipped with an inner product.

1.2.6 Sobolev Spaces

The Sobolev space is a functional space, and it could have been listed in the previous subsection. However, we have placed it in a separate subsection because of its contents and role in the text.

On many occasions involving differential operators, it is convenient to incorporate the L^p norms of the derivative of a function into a Banach norm. Consider the functions in the class $C^\infty(\Omega)$. For any number $p \geq 1$ and number $s \geq 0$, let us take the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{s,p} = \left\{ \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^p}^p \right\}^{1/p}. \tag{1.2.5}$$

The resulting Banach space is called the Sobolev space $W^{s,p}(\Omega)$. For $p = 2$ we denote $W^s(\Omega) = W^{s,2}(\Omega)$, which is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{s,2} = \sum_{|\alpha| \leq s} \int_\Omega \overline{D^\alpha u} \cdot D^\alpha v \, dx.$$

Sometimes $W^s(R)$ is also denoted as $H^s(R)$. Note that the differentiation in (1.2.5) can be of a noninteger.

Recall that the Fourier transform of the derivative $f'(t)$ is $i\omega \hat{f}(\omega)$. The Plancherel–Parseval formula proves that $f'(t) \in L^2(R)$ if

$$\int_{-\infty}^{\infty} |f'(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^2 |\hat{f}(\omega)|^2 < +\infty.$$

This expression can be generalized for any $s > 0$,

$$\int_{-\infty}^{\infty} |\omega|^{2s} |\hat{f}(\omega)|^2 \, d\omega < +\infty$$

if $f \in L^2(R)$ is s times differentiable.

Considering the summation nature of (1.2.5), we can write the more precise expression of Sobolev space in the Fourier domain as

$$\int_{-\infty}^{\infty} (1 + \omega^2)^s |f(\omega)|^2 \, d\omega < +\infty.$$

For $s > n + \frac{1}{2}$, f is n times continuously differentiable. The Sobolev space H^α, $\alpha \in R$ consists of functions $f(t) \in S'$ such that

$$\int_{-\infty}^{\infty} \hat{f}(\omega)(1 + \omega^2)^\alpha \, d\omega < \infty.$$

For $\alpha = 0$, the H^α reduces to $L^2(R)$. For $\alpha = 1, 2, \ldots, H^\alpha$ is composed of ordinary $L^2(R)$ functions that are $(\alpha - 1)$ times differentiable and whose αth derivative are in $L^2(R)$. For $\alpha = -1, -2, \ldots,$ H^α contains the $-\alpha$th derivatives of $L^2(R)$ and all distributions with point support of order $< \alpha$.

It can be seen $H^\alpha \supset H^\beta$ when $\alpha > \beta$. The inner product of $f, g \in H^\alpha$ is

$$\langle f, g\rangle_\alpha = \frac{1}{2\pi}\int \overline{\hat{f}(\omega)}\hat{g}(\omega)(1+\omega^2)^\alpha \, d\omega$$

and is complete with respect to this inner product. Therefore it is a Hilbert space.

1.2.7 Bases in Hilbert Space *H*

Orthonormal Basis. A sequence $\{f_n\}_{n\in N}$ in a Hilbert space H is orthonormal if

$$\langle f_m, f_n\rangle = \delta_{m,n}.$$

If for $f \in H$ there exist α_n such that

$$\lim_{N\to\infty} \|f - \sum_{n=0}^{N} \alpha_n f_n\| = 0,$$

then $\{f_n\}_{n\in N}$ is called an orthogonal basis of H.

For an orthonormal basis we require $\|f_n\| = 1$. A Hilbert space that admits an orthogonal basis is said to be separable. The norm of $f \in H$ is

$$\|f\|^2 = \sum_{n=0}^{\infty} |\langle f, f_n\rangle|^2$$

Riesz Basis. Let $\{f_n\}$ be linear independent and complete in $L^2(a, b)$, meaning that the closed linear span of $\{f_n\}$ is $L^2(a, b)$. The set is called a Riesz basis if there exist $A > 0$ and $B > 0$ such that

$$A\sum_i |c_i|^2 \le \|\sum_i c_i f_i\|^2 \le B\sum_i |c_i|^2 \tag{1.2.6}$$

for each sequence $\{c_i\}$ of complex numbers. The Riesz representation theorem guarantees the existence of the dual $\{\tilde{f}_n\}$ in $L^2(a, b)$ such that:

(1) $\{\tilde{f}_n\}$ is the unique biorthogonal sequence to $\{f_n\}$; namely $\langle f_m, \tilde{f}_n\rangle = \delta_{m,n}$.
(2) If $\{c_n\} \in \ell^2$, then $\sum_n c_n f_n$ converges in $L^2(a, b)$.
(3) For each $f \in L^2(a, b)$, $\{\langle f, \tilde{f}_n\rangle\} \in \ell^2$.
(4) For each $f \in L^2(a, b)$,

$$f = \sum_{i=0}^{\infty}\langle f, \tilde{f}_i\rangle f_i = \sum_{i=0}^{\infty}\langle f, f_i\rangle \tilde{f}_i.$$

A Riesz basis of a separable Hilbert space H is a basis that is close to being orthogonal. The right inequality in (1.2.6) is essential. It prevents the expansion from blowing up. The left inequality in (1.2.6) is important too, since it ensures the existence of the inverse.

1.2.8 Linear Operators

In computational electromagnetics, the method of moments and finite element method are based on linear operations. An operator T from a Hilbert space H_1 to another Hilbert space H_2 is linear if

$$\forall \alpha_1, \alpha_2 \in \mathcal{C},\ \forall f_1, f_2 \in H_1, \quad T(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 T(f_1) + \alpha_2 T(f_2).$$

Sup Norm. The sup operator norm of T is defined as

$$\|T\|_S = \sup_{f \in H_1} \frac{\|Tf\|}{\|f\|}. \tag{1.2.7}$$

If this norm is finite, then T is continuous; namely $\|Tf_1 - Tf_2\|$ becomes arbitrarily small if $\|f_1 - f_2\|$ is sufficiently small.

Adjoint. The adjoint of T is the operator T^a from H_2 to H_1 such that for any $f_1 \in H_1$ and $f_2 \in H_2$

$$\langle Tf_1, f_2 \rangle = \langle f_1, T^a f_2 \rangle.$$

When T is defined from H into itself, it is self-adjoint if $T = T^a$. A nonzero vector $f \in H$ is a called an *eigenvector* if there exists an *eigenvalue* $\lambda \in \mathcal{C}$ such that

$$Tf = \lambda f.$$

In a finite-dimensional Hilbert space, meaning that Euclidean space, a self-adjoint operator is always diagonalized by an orthogonal basis $\{e_n\}_{0 \le n < N}$ of eigenvectors

$$Te_n = \lambda_n e_n.$$

For a self-adjoint operator T, the eigenvalues λ_n are real, and for any $f \in H$

$$Tf = \sum_{n=0}^{N-1} \langle Tf, e_n \rangle e_n = \sum_{n=0}^{N-1} \lambda_n \langle f, e_n \rangle e_n.$$

In an infinite-dimensional Hilbert space, the previous result can be generalized in terms of the spectrum of the operator, which must be manipulated with caution.

Orthogonal Projector. Let V be a subspace of H. A *projector* P_V on V is a linear operator that satisfies $\forall f \in H, \quad P_V f \in V$ and $\forall f \in V, \quad P_V u = f$.

The projector P_V is *orthogonal* if

$$\forall f \in H, \forall g \in V, \quad \langle f - P_V f, g\rangle = 0.$$

The following properties are often used in the text:

Property 1. If P_V is *a projector* on V, then the following statements are equivalent:

(1) P_V is *orthogonal.*
(2) P_V is *self-adjoint.*
(3) $\|P_V\|_S = 1$.
(4) $\forall f \in H$, $\|f - P_V f\| = \min_{g\in \mathbf{v}} \|f - g\|$.

If $\{e_n\}_{n\in N}$ is an orthogonal basis of V, then

$$P_V f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n\rangle}{\|e_n\|^2} e_n.$$

If $\{e_n\}_{n\in N}$ is a Riesz basis of V and $\{\tilde{e}_n\}_{n\in N}$ is the biorthogonal basis, then

$$P_V f = \sum_{n=0}^{+\infty}\langle f, e_n\rangle\tilde{e}_n = \sum_{n=0}^{+\infty}\langle f, \tilde{e}_n\rangle e_n.$$

Density and Limit. A space V is *dense* in H if for any $f \in H$ there exist $\{f_m\}_{m\in N}$ with $f_m \in V$ such that

$$\lim_{m\to+\infty} \|f - f_m\| = 0.$$

Let $\{T_n\}_{n\in N}$ be a sequence of linear operators from H to H. Such a sequence *converges* weakly to a linear operator T_∞ if

$$\forall f \in H, \quad \lim_{n\to+\infty} \|T_n f - T_\infty f\| = 0.$$

To find the limit of operators it is preferable to work in a well chosen subspace $V \subset H$ which is dense. The density and limit are justified by the property below.

Property 2 (Density). Let V be a *dense subspace* of H. Suppose that there exists C such that $\|T_n\|_S \leq C$ for all $n \in N$. If

$$\forall f \in V, \quad \lim_{n\to+\infty} \|T_n f - T_\infty f\| = 0,$$

then

$$\forall f \in H, \quad \lim_{n\to+\infty} \|T_n f - T_\infty f\| = 0.$$

For numerical computations, an operator is often discretized into a matrix. Only then digital computers can be utilized.

Norm of a Matrix. For a matrix $A \in R^{n\times n}$, the norm of A is defined, similarly to (1.2.7), as

$$\| A \| = \max_{x\neq 0} \left\{ \frac{\| Ax \|}{\| x \|} \right\}.$$

In particular, the commonly used norms are as follows:

(1) The column norm (ℓ^1 norm)

$$\| A \|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|.$$

(2) The row norm (ℓ^∞ norm)

$$\| A \|_\infty = \max_i \{\|a_{i\cdot}\|_1\} = \max_i \sum_{j=1}^{n} |a_{i,j}|.$$

(3) The spectral norm (ℓ^2 norm)

$$\|A\|_2 = (\lambda_{A^T A})^{1/2},$$

where $\lambda_{A^T A}$ is the maximum eigenvalue of $A^T A$.

(4) The Frobenius norm

$$\|A\|_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{i,j}|^2 \right)^{1/2} = [\mathrm{tr}\,\{A^T A\}]^{1/2}.$$

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CHAPTER TWO

Intuitive Introduction to Wavelets

2.1 TECHNICAL HISTORY AND BACKGROUND

The first questions from those curious about wavelets are: What is a wavelet? Who invented wavelets? What can one gain by using wavelets?

2.1.1 Historical Development

Wavelets are sometimes referred to as the twentieth-century Fourier analysis. Wavelets exploit the multiresolution analysis just like microscopes do in microbiology. The genesis of wavelets began in 1910 when A. Haar proposed the staircase approximation to approximate a function, using the piecewise constants now called the Haar wavelets [1]. Afterward many mathematicians, physicists, and engineers made contributions to the development of wavelets:

- Paley–Littlewood proposed dyadic frequency grouping in 1938 [2].
- Shannon derived sampling theory in 1948 [3].
- Calderon employed atomic decomposition of distributions in parabolic H^p spaces in 1977 [4].
- Stromberg improved the Haar systems in 1981 [5].
- Grossman and Morlet decomposed the Hardy functions into square integrable wavelets for seismic signal analysis in 1984 [6].
- Meyer constructed orthogonal basis in L^2 with dilation and translation of a smooth function in 1986 [7].

- Mallat introduced the multiresolution analysis (MRA) in 1988 and unified the individual constructions of wavelets by Stromberg, Battle–Lemarie, and Meyer [8].
- Daubechies first constructed compactly supported orthogonal wavelet systems in 1987 [9].

2.1.2 When Do Wavelets Work?

Most of the data representing physical problems that we are modeling are not totally random but have a certain correlation structure. The correlation is local in time (spatial domain) and frequency (spectral domain). We should approximate these data sets with building blocks that possess both time and frequency localization. Such building blocks will be able to reveal the intrinsic correlation structure of the data, resulting in powerful approximation qualities: only a small number of building blocks can accurately represent the data. In electromagnetics the compactly supported (strictly localized in space) wavelets may be used as basis functions. These wavelets, by the Heisenberg uncertainty principle (or by Fourier analysis), cannot have strictly finite spectrum, but they can be approximately localized in spectrum. If most of their spectral components are beyond the visible region, for example, $\kappa_x > k_0$, they will produce little radiation, resulting in a sparse impedance matrix in the method of moments.

The previous observations may be generalized and described more precisely:

(1) Wavelets and their duals are local in space and spectrum. Some wavelets are even compactly supported, meaning strictly local in space (e.g., Daubechies and Coifman wavelets) or strictly local in spectrum (e.g., Meyer wavelets). Spatial localization implies that most of the energy of a wavelet is confined to a finite interval. In general, we prefer fast (exponential or inverse polynomial) decay away from the center of mass of the function. The frequency localization means band limit. The decay toward high frequencies corresponds to the smoothness of the wavelets; the smoother the function is, the faster the decay. If the decay is exponential, the function is infinitely many times differentiable. The decay toward low frequencies corresponds to the number of vanishing polynomial moments of the wavelet. Because of the time-frequency localization of wavelets, efficient representation can be obtained. The idea of frequency localization in terms of smoothness and vanishing moments may generalize the concept of "frequency localization" to a manifold, where the Fourier transform is not available.

(2) Wavelet series converge uniformly for all continuous functions, while Fourier series do not. In electromagnetics, the fields are often discontinuous across material boundaries. For piecewise smooth functions, Fourier-based methods give very slow convergence, for example, $\alpha = 1$, while nonlinear (i.e. with truncation) wavelet-based methods, exhibit fast convergence [10], for example, $\alpha \geq 2$, where α is the convergence rate defined by $\|f - f_M\| = O(M^{-\alpha})$

and the M-term approximate of f is given by

$$f_M = \sum_{\lambda \in \Lambda_M} c_\lambda \psi_\lambda. \tag{2.1.1}$$

(3) Wavelets belong to the class of orthogonal bases that are continuous and problem independent. As such, they are more suitable for developing systematic algorithms for general purpose computations. In contrast, the pulse bases, although orthogonal and compact in space, are not smooth. Indeed, they are discontinuous and are not localized in the spectral domain. On the other hand, Chebyshev, Hermite, Legendre, and Bessel polynomials are orthogonal but not localized in space within the domain (in comparison with intervallic and periodic wavelets). Shannon's *sinc* functions are localized in the transform domain but not in the original domain. The eigenmode expansion method is based on orthogonal expansion, but is problem dependent and works only for limited specific cases (e.g., rectangular, circular waveguides) [11].

(4) Wavelets decompose and reconstruct functions effectively due to the multiresolution analysis (MRA), that is, the passing from one scale to either a coarser or a finer scale efficiently. The MRA provides the fast wavelet transform, which allows conversion between a function f and its wavelet coefficients c with linear or linear-logarithmic complexity.

2.1.3 A Wave Is a Wave but What Is a Wavelet?

The title of this section is a note in the June 1994 issue of *IEEE Antennas and Propagation Magazine* from Professor Leopold B. Felson. Wavelet is literally translated from the French word *ondelette*, meaning small wave.

Wavelets are a topic of considerable interest in applied mathematics. One may use wavelets to decompose data, functions, and operators into different frequency components, and then study each component with a "resolution" level that matches the "scale" of the particular component. This "multiresolution" technique outperforms the Fourier analysis in such a way that both time domain and frequency domain information can be preserved. In a loose sense, one may say that the wavelet transform performs the optimized sampling. In contrast to the wavelet transform, the windowed Fourier transform oversamples the object under investigation, with respect to the Nyquist sampling criterion. Again, in a loose sense, one can say that wavelets decompose and compress data, images, and functions with good basis systems to reach high efficiency or sparseness. A key point to understand about wavelets is the introduction of both the dilation (frequency information) and translation (local time information).

Wavelets have been applied with great success to engineering problems, including signal processing, data compression, pattern recognition, target identification, computational graphics, and fluid dynamics. Recently wavelets have also been used in boundary value problems because they permit the accurate representation of a variety of operators without redundancy.

2.2 WHAT CAN WAVELETS DO IN ELECTROMAGNETICS AND DEVICE MODELING?

2.2.1 Potential Benefits of Using Wavelets

Owing to their ability to represent local high-frequency components with local basis elements, wavelets can be employed in a consistent and straightforward way. It is well known to the electromagnetic modeling community that the finite element method (FEM) is a technique that results in sparse matrices amenable to efficient numerical solutions. For the FEM the solution times tend to increase by $n \log(n)$, where $n \sim N^3$, with N being the number of points in one dimension. In using surface integral equations, implemented by the method of moments (MoM), the solution times have been demonstrated to increase by M^3, where $M \sim N^2$. It is obvious that N^2 is much smaller than N^3, and that therefore the MoM deals with many fewer unknowns than the FEM. Unfortunately, the matrix from the MoM is dense. The corresponding computational cost, using the direct solver, is on the order of $O(n^3)$, where $n \sim N^2$. It is clear that the solution of dense complex matrices is prohibitively expensive, especially for electrically large problems.

Integral operators are represented in a classical basis as a dense matrix. In contrast, wavelets can be seen as a quasi-diagonalizing basis for a wider class of integral operators. The "quasi" is necessary because the resulting wavelet expansion of integral operators is not truly diagonal. Instead, it has a peculiar palm pattern. This palm-type sparse structure represents an approximation of the original integral operator to arbitrary precision. It was reported that wavelet-based impedance matrices contain 90 to 99% zero entries. It has been shown by mathematicians that the solution of a wide range of integral equations can be transformed, using wavelets, from a direct procedure requiring order $O(n^3)$ operations to that requiring only order $O(n)$ [12]. In recent years, wavelets have been applied to electromagnetics and semiconductor device modeling for several purposes:

(1) To solve surface integral equations (SIE) originating from scattering, antenna, packaging and EMC (electromagnetic compatibility) problems, where very sparse impedance matrices have been obtained. It was reported that the wavelet scheme reduces the two-norm condition number of the MoM matrix by almost one order of magnitude [13].

(2) To improve the finite difference time domain (FDTD) algorithms in terms of convergence and numerical dispersion using Daubechies sampling biorthogonal time domain method (SBTD).

(3) To improve the convergence of the finite element method (FEM) using multiwavelets as basis functions.

(4) To solve nonlinear partial differential equations (PDEs) via the collocation method, in which the nonlinear terms in the PDEs are treated in the physical space while the derivatives are computed in the wavelet space [14].

(5) To model nonlinear semiconductor devices, where the finite difference method is implemented on the adaptive mesh, based on the interpolating wavelets and sparse point representation.

Some fascinating features of wavelets in the aforementioned applications are as follows:

(1) For the finite difference time domain (FDTD) method, numerical dispersion has been improved greatly. By imposing the Daubechies wavelet-based sampling function and its dual reproducing kernel, the SBTD requires much coarser mesh size in comparison with the Yee-FDTD while achieving the same precision. For a 3D resonator problem, the SBTD improves the CPU time by a factor of 13, and memory by 64. Material inhomogeneity and boundary conditions can be easily incorporated [15].
(2) For the finite element method (FEM), the multiwavelet basis functions are in C^1. At the node/edge, they can match not only the function but also its derivatives, yielding faster convergence than the traditional high-order FEM. For a partially loaded waveguide, the improvement of multiwavelet FEM over linear basis EEM exceeds 435 in CPU time reduction [16].
(3) For packaging and interconnects, the wavelet-based MoM speeds up parasitic parameter extraction by 1000 [17].
(4) Often in semiconductor device modeling, a small part of the computational interval or domain contains most of the activity, and the representation must have high resolution there. In the rest of the domain such high resolution is a high-cost waste. Various adaptive mesh techniques have been developed to address this issue. However, they often suffer accuracy problems in the application of operators, multiplication of functions, and so on. Wavelets offer promise in providing a systematic, consistent and simple adaptive framework. In the simulation of a 2D abrupt diode, the potential distribution was computed using wavelets to achieve a precision of 1.6% with 423 nodes. The same structure was simulated by a commercial package ATLAS, and 1756 triangles were used to reach a 5% precision [18].
(5) Coifman wavelets allow the derivation of a single-point quadrature of precision $O(h^5)$, which reduces the impedance filling process from $O(n^2)$ to $O(n)$.

2.2.2 Limitations and Future Direction of Wavelets

Wavelets are relatively new and are still in their infancy. Despite the advantages and beneficial features mentioned above, there are difficulties and problems associated in using wavelets for EM modeling.

Classical wavelets are defined on the real line, while many real world problems are in the finite domain. Periodic and intervallic wavelets have provided part of the solution, but they have also increased the complexity of the algorithm. Multiwavelets seem to be very promising in solving problems on intervals because of their orthogonality and interpolating properties.

The problems and difficulties encountered in practical fields have stimulated the interest of mathematicians. In recent years mathematicians have constructed wavelets on closed sets of the real line, satisfying certain types of boundary conditions. They have also studied wavelets of increasing order in arbitrary dimensions [19], wavelets on irregular point sets [20], and wavelets on curved surfaces as in the case of spherical wavelets [21].

2.3 THE HAAR WAVELETS AND MULTIRESOLUTION ANALYSIS

One of the most important properties of wavelets is the multiresolution analysis (MRA). Without losing generality, we discuss the MRA through the Haar wavelets. The Haar is the simplest wavelet system that can be studied immediately without any prerequisite. Later we will pass these conclusions on to other orthogonal wavelets. Therefore mathematical proofs are bypassed.

The Haar scaling functions (or scalets) are defined as

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.1}$$

The Haar mother wavelets (or wavelets) are defined as

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.2}$$

These two functions are sketched in Fig. 2.1. In the rest of the book, we will refer to mother wavelets as wavelets and scaling functions as scalets, in order to emphasize their roles as counterparts of wavelets. Notice that the term "wavelets" has a dual meaning. Depending on the context, wavelet can mean the wavelet or both the scalet and wavelet.

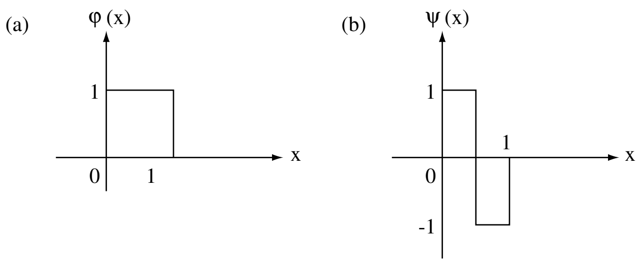

FIGURE 2.1 Haar (*a*) scalet and (*b*) wavelet.