An Introduction to Parametric Digital Filters and Oscillators

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To my wife Irina
and our sons
Pavel, Alexei and Andrei

Contents

Preface

Digital signal processing (DSP) does not require any special advertisements. Since the 1960s, it has become one of the most intensive fields of study in electronics-related science, and since the 1980s, owing to the extensive progress in integrated circuits technology, it has been an inseparable part of modern electronic systems. However, among the numerous DSP publications on algorithms, approaches, technical solutions, and so on, there is apparently no book on library shelves that is dedicated to linear non-adaptive time-variant digital filters. The lack of such a book is a deterrent to developing much broader engineering applications of these systems.

Different aspects of time-variant digital filters, or broader systems, have been studied for many years. Publications dedicated to this subject belong to different authors, and are spread over years and across journals. However, in spite of the many interesting and useful features of such systems, there are no systematic publications, monographs, or textbooks dedicated to filters with time-varying parameters or more complex systems based on these filters. The objective of this book is to present an appropriate introduction to theory and practice of one of the subclasses of time-varying digital systems: parametric digital filters and oscillators. The word *parametric* adopted in this book came from analog systems with periodically time-varying parameters; for example, the RLC resonator with varying capacitor [1]. This book starts with an analysis of discrete systems with parameters varying according to arbitrary laws, while the core of the book is dedicated to digital parametric filters and oscillators, which are the systems with periodically time-varying coefficients. In the general case, coefficient variation laws are arbitrary but specified beforehand, regardless of the input process. This distinguishes the discussed systems from adaptive filters [2]. This book does not cover filters with an essentially varying sampling rate $nT + \delta T(n)$ and $\delta T(n) \geq T$, which belong to the subclass of multi-rate filters [3] and also, in many instances, belong to the class of time-variant systems [4].

Thus, we will study digital systems described by the linear difference equation with time-varying parameters:

$$\sum_{k=0}^{K_1} a_k(n) \cdot y(n-k) = \sum_{k=0}^{K_2} b_k(n) \cdot x(n-k)$$

where $x(n)$ and $y(n)$ are input and output signals respectively; $n = 0, 1, \ldots$ is the time instant nT (T is the sampling interval); $a_k(n)$ and $b_k(n)$ are time-varying coefficients; and $a_0(n) \neq 0$ for any n.

Choosing an appropriate law of parameter variation in infinite impulse response (IIR) systems allows them to operate in *filtering, frequency conversion* or *parametric oscillating modes.* The latter mode has not been previously discussed in the literature except in the author's publications. In the main text, in many cases the word "filter" will describe all these systems. There will not be a focus on how to build these systems. The presented algorithms for time-variant systems will be appropriate for universal computers, microprocessors, specially developed hardware or DSP boards. For us, these will all be time-variant systems or filters.

Time-variant systems demonstrate some essential peculiarities in comparison with the traditional digital time-invariant filters. Even very small variations in parameters can change the characteristics of filters dramatically. Distinctive features of these systems are interesting from the circuit theory point of view and also have practical applications. Looking at this problem a little bit philosophically, we can regard the variation of parameters in time as offering new degrees of freedom in system design. Readers will find numerous examples within this book of how these extra degrees of freedom influence filter characteristics.

But, first let us look at an example that is very far from the field of digital systems. This example shows how it can be important to add an extra degree of freedom when attempting to solve a problem.

So, there are problems that have no solutions within N × D space, but have solutions within (N + K) × D space or have better solutions within (N + K) × D space, or have more cost-effective solutions and so on.

Comparison of the difference equation describing time-invariant filters

$$\sum_{k=0}^{K_1} a_k \cdot y(n-k) = \sum_{k=0}^{K_2} b_k \cdot x(n-k)$$

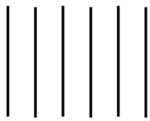
with the difference equation describing time-variant filters shows that the latter has extra degrees of freedom owing to the time dependence of coefficients. How these new degrees of freedom can be used will be discussed in the main text. The author hopes that on the basis of this information, researchers and engineers will be able to develop many new applications for time-variant digital systems.

In the book, only two algorithms of time-variant systems are discussed in detail: frequency filters that are, in some instances, equivalents of linear time-invariant (LTI) filters, and parametric oscillators. Of course, these are not the only possible types of linear time-variant (LTV) system applications. LTV systems are optimal, for example, for cyclo-stationary signals processing in communication systems [5, 6]. LTV discrete systems (DSs) can be used for spectrum [7] and image scrambling [8], image transmission [9], systems identification [10], TDM/FDM conversion [11, 12] and for many other useful applications.

The last but not the least group of LTV algorithms are two-dimensional time-variant filters for image processing, which are now the focus of much research. They include periodically time-varying filters [13] as well as more general systems and,

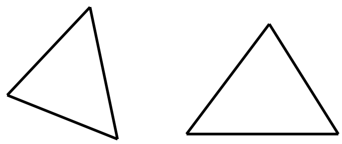
PUZZLE

You have six matches.



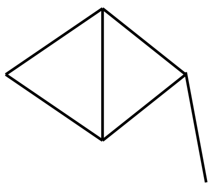
How do you build four triangles
using only these six matches?

The first attempt:



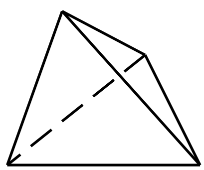
Six matches are used but only two triangles are built

The second attempt:



Two and a half triangles are ready but all matches have been used.

Keep going…



The solution is a pyramid and an extra degree of freedom is the third dimension.

in particular, time-variant filtering based on Gabor transform [14, 15]. Traditionally, one-dimensional filtering theory has generally been the basis for multidimensional signal processing. Therefore, this book can also be used as an introduction to two-dimensional LTV filtering.

As follows from the discussion above, LTV systems represent a rather broad class of systems and algorithms for signal and image processing. This book does not pretend to cover all aspects of LTV DS analysis and synthesis as well as application of time-varying algorithms in signal processing. Following the advice of the Russian folk philosopher Kozma Prutkoff that

"*. . . it is impossible to envelop the boundless . . .*"

this book is necessarily restricted in its contents. However, the author's expectation is that the book will initiate a new wave of interest in this class of systems, particularly in the engineering community.

The book contains seven chapters. There are no cross-references between the introduction and the main text, allowing the main text to be read independently of the introduction. When the first draft of the main text was ready, the author gave it to some postgraduate students to study. However, it took an unexpectedly long time for students to complete their reading of the book. After discussions with these students about how to make the book easier to read, the introduction was added. It is designed to help the reader understand the main text without requiring other special materials. The introductory chapter concisely explains the general problems of digital signals, filtering and methods of system analysis.

The introduction is not intended to substitute for numerous wonderful textbooks dedicated to digital systems and signals [16–18]. So, if readers feel confident about their knowledge of digital signals and systems they can read the book starting from the main text. Alternatively, the introduction may serve to refresh the reader's knowledge of the signals and systems basics.

This book is written, first of all, for graduate specialists in signal processing and related specialties, as well as for PhD students. Other students, for example, those engaged in final year thesis preparation, may also find it useful.

Any preface assumes some historical reference to the subject. For me, the story of this subject started when I first read the paper of reference [19]. I then started to work in this area with my PhD students. Much later I had the privilege of spending a term in Cambridge University with a world-class signal-processing group led by Prof. Peter Rayner. Some early research done by this group was also dedicated to time-variant signal processing [11, 20].

Most of the author's papers dedicated to parametric systems have been published in Russian. It is difficult to translate properly even the title of these journals. Some information regarding these papers can be found in [21].

My former postgraduate students, V. Bets, V. Sizov, I. Rogozkin, L. Donskoi, P.-J. Picot, have contributed a lot in the area covered by the book. Moreover, with the permission of V. Sizov, there are some direct adoptions from his thesis; in particular, examples of time-varying filters.

The book is also a good place to thank my former PhD supervisor and later my colleague for many years, Prof. D. Nezlin, for his contribution to my development as a scientist.

Behind any book there is a big job in manuscript preparation. I want to thank Carol Booth who helped me with this.

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1

Introduction: Basis of Discrete Signals and Digital Filters

The theory and practice of digital signal processing (DSP) are currently in a mature stage. It is difficult to imagine any modern electronic system without wide application of DSP and, in particular, linear time-invariant algorithms for filtering, equalization, characteristic correction and so on.

The major goal of this chapter is to introduce the theoretical basis of discrete signals and time-invariant digital systems to help readers more easily understand the main text dedicated to time-variant systems and to minimize the necessity to consult other texts while reading this book. This introduction provides a superficial overview of DSP concepts: sampling and quantization; impulse and frequency responses; Fourier, Laplace and z-transforms; system stability and causality and finite and infinite impulse response (IIR) digital filters (DFs). For those familiar with DSP and related subjects, this introduction will help refresh their knowledge. For those who are unfamiliar, this chapter can be used as the first stage of study of discrete signals and systems. Of course, this introduction does not and cannot replace special literature and textbooks dedicated to DSP problems. Among the latest textbooks in this area, the author recommends Reference [1].

1.1 DISCRETE SIGNALS AND SYSTEMS

Most signals used in information systems are similar to analog processes. In the general case they are functions of continuous time. Digital filters belong to the group of discrete systems of signal processing, which operate with discrete input processes. Thus, an analog input signal is represented by discrete samples obtained in time moments proportional to the sampling interval T. An analog waveform can be transformed into an appropriate discrete signal without information losses if sampling frequency f_s is determined as

$$f_s = \frac{\omega_s}{2\pi} = \frac{1}{T} \geq 2 f_{o\,\max} \qquad (1.1)$$

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This corresponds to the Nyquist criteria, that is, the sampling frequency is at least two times higher than the highest frequency in the signal spectrum $f_{o\max}$ [2]. In discrete signal analysis, frequency, as a rule, is represented as a normalized frequency $\omega = \omega_a T = \omega_a / f_s$, where $\omega_a = 2\pi f_a$ is a frequency of an analog (continuous) signal.

To form a digital signal from a discrete signal, the amplitude is represented as a binary code. The device that quantizes the signal is called an *analog–digital converter* (ADC). The number of bits in signal representation depends on the system's applications and in practice, varies in a band from 1 to 16. The most widely used ADCs have 8 to 12 bits.

The analysis of digital systems is similar to the analysis of analog systems and is based on the comparison of signals at the system's input and output. In this chapter, digital signals and systems will be considered with the assumption that the number of bits in ADCs is large enough and that quantization effects are negligible. In other words, we make digital signals and systems equivalent to discrete signals and systems. If necessary, a quantization effect can be taken into account by adding some quantization noise to signal. In conventional ADCs, in the first approximation, this noise has uniformly distributed amplitude with zero mean value and its power can be calculated by [1] $\sigma_{qn}^2 = \Delta^2/12$, where Δ is the quantization level. This noise also has near uniform power spectral density over the band $|f| \leq f_s/2$.

Signal-to-quantization noise ratio (S/N_{qn}) can be evaluated as $S/N_{qn} \approx 6.02 B_{its} + 4.77 - 20\log(A_p/\sigma_S)$ (dB), where B_{its} is the number of bits representing an input signal, σ_S is the rms value of the input waveform and A_p is the ADC peak design level of the quantizer. For example, if an input signal is a sinusoidal waveform $(S/N_{qn}) \approx 6.02 B_{its} + 1.7$ (dB). Continuous linear systems are fully characterized by their impulse response $h(t)$. The impulse response is an output system reaction to the input signal, described by the δ-function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{1.2}$$

and

$$y(t) = \int_0^t x(t-\lambda) h(\lambda)\, \mathrm{d}\lambda \tag{1.3}$$

where $x(t)$ and $y(t)$ are input and output signals of the system respectively, and $x(t) = 0$ for $t < 0$.

For a discrete system, continuous time t should be replaced by discrete time $t = nT$ and $\lambda = mT$, and integration is replaced by summation

$$y(nT) = \sum_{m=0}^{n} x(nT - mT) \cdot h(mT) \cdot T \tag{1.4}$$

Thus, the first step of digital system analysis is the representation of an analog signal $x(t)$ by a discrete equivalent $x(nT)$. The second step is the representation of $h(t)$ by its discrete equivalent $h(mT)$.

1.2 DISCRETE SIGNALS

1.2.1 Time-Domain Representation for Discrete Signals

In the general case, discrete signals can be described in discrete time moment nT as well as in continuous time. For the analysis of discrete systems, signals description in discrete time is most popular, namely, nT. The sampling period T is often omitted and the signal at the moment nT is described as $x(n) = x(nT)$.

Some examples of discrete signal descriptions and their plots are given below.

1. Sinusoidal sequence: $x(nT) \equiv x(n) = \sin(\omega nT) \equiv \sin(\omega n)$ (Fig. 1.1)
2. Linear sequence: $x(nT) \equiv x(n) = n$ (Fig. 1.2)
3. Unit sample sequence (impulse): $x_i(n - m) = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$ (Fig. 1.3)
4. Unit step sequence: $x_s(n - m) = \begin{cases} 1 & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases}$ (Fig. 1.4)

Unit steps and unit impulses are widely used as test signals to analyse discrete systems. It is sometimes convenient to represent function $x_s(n)$ as a function $x_i(n)$: $x_s(n - k) = \sum_{m=0}^{\infty} x_i(n - k - m)$.

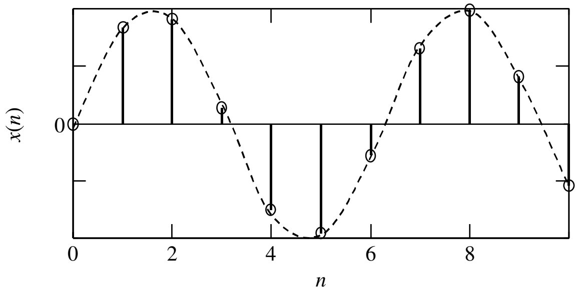

Figure 1.1 Discrete function $x(n) = \sin(\omega n)$

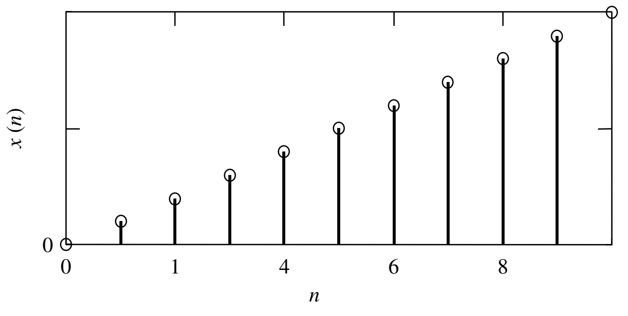

Figure 1.2 Discrete function $x(n) = n$

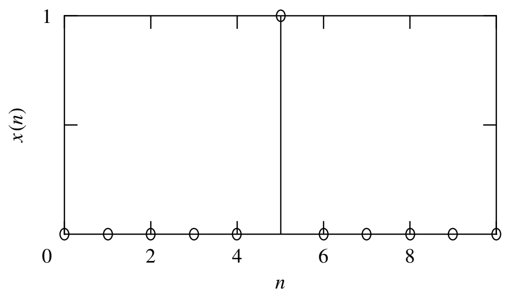

Figure 1.3 Unit sample, $m = 5$

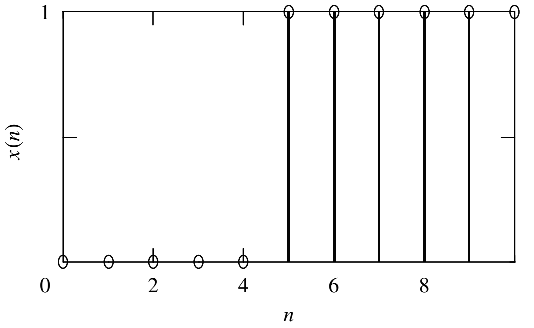

Figure 1.4 Unit step, $m = 5$

1.2.2 Presentation of Discrete Signals by Fourier Transform

Like analog signals, discrete signals can be represented and analysed in frequency domain. Spectral analysis is based on Fourier transform [2]. To apply Fourier transform to discrete signals, they have to be represented in continuous time

$$x(n) = x(nT) = x_d(t) = x(t) \cdot v(t) \tag{1.5}$$

where $x_d(t)$ is a discrete function represented in continuous time, $x(t)$ is the initial analog function (e.g., $x(n) = \sin(\omega n) \Leftrightarrow x(t) = \sin(\omega_a t)$) and $v(t)$ is a periodical sequence of δ-functions (see Fig. 1.5a) with period T

$$v(t) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right), \quad n = 0, 1, 2, \ldots \tag{1.6}$$

Note that the δ-function possesses some properties that will be used later

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \tag{1.7}$$

$$\int_{-\infty}^{\infty} F(t)\delta(t - t_0)\,\mathrm{d}t = F(t_0) \tag{1.8}$$

where $F(t)$ is an arbitrary function. Thus, discrete function $x(n)$ in continuous time can be represented by

$$x(n) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right) \tag{1.9}$$

As was discussed earlier, a discrete function can be obtained from an appropriate analog function by discretization. But, from a practical point of view, δ-function is an abstract notion. So, for practical applications, it is more useful to consider an impulse sequence with a unit amplitude and limited duration τ (Fig. 1.5b) as a periodical sampling function:

$$v_\tau(t) = \begin{cases} 1 & \text{for } |t - nT| \le \dfrac{\tau}{2} \\ 0 & \text{for } |t - nT| > \dfrac{\tau}{2} \end{cases} \tag{1.10}$$

Then the discrete signal takes the form

$$x(n) = x(t) \cdot v_\tau(t) \tag{1.11}$$

To evaluate a spectrum of this discrete function, let us consider known expressions for a continuous waveform s(t) spectrum [2]

$$\overline{S}(\omega_a) = \int_{-\infty}^{\infty} s(t) \exp(-\mathrm{j}\omega_a t)\, \mathrm{d}t \tag{1.12}$$

where $\overline{(*)}$ denotes a complex function. We use equation (1.9) to calculate the spectrum of the discrete signal $x(n)$. Assume that $x(n) = 0$ for $n < 0$ and introduce $x(n)$ via its continuous time equivalent

$$\overline{X}_d(\omega_a) = \int_0^{\infty} x(t) \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right) \exp(-\mathrm{j}\omega_a t)\, \mathrm{d}t$$

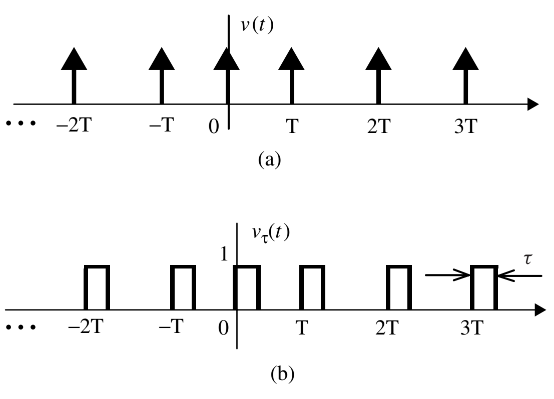

Figure 1.5 Sample functions: (a) ideal and (b) real

$$= \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} x(t)\delta\left(\frac{t}{T} - n\right) \exp(-j\omega_a t)\, dt$$

$$= T \sum_{n=-\infty}^{\infty} x(nT) \exp(-j\omega_a nt) \tag{1.13}$$

As seen from equation (1.13), the sampling period T is a scale factor, and in some literature, it is omitted. So, the spectrum of a discrete signal is generally a complex value and is a function of the analog frequency ω_a. However, in many cases, it is more convenient to represent this spectrum as a function of normalized frequency $\omega = \omega_a T$ or

$$\overline{X}_d(\omega) \equiv X(\omega) = \sum_{n=0}^{\infty} x(n) \exp(-j\omega n) \tag{1.14}$$

for the case $x(n) = 0$ when $n < 0$. In spectrum descriptions, complexity notation $\overline{(*)}$ is also often omitted, taking into account that the spectrum, in general, is a complex value.

From expression (1.13), it follows that the discrete signal spectrum is periodical with period ω_s. This important property can be described more accurately

$$\overline{X}_d(\omega_a + k\omega_s) = T \sum_{n=0}^{\infty} x(nT) \exp[-j(\omega_a + k\omega_s)nT]$$

$$= T \sum_{n=0}^{\infty} x(nT) \exp(-j\omega_a nT) \cdot \exp(-jk\omega_s nT) \tag{1.15}$$

However,

$$\exp(-jk\omega_s nT) = \left(-jk\frac{2\pi}{T}nT\right) = 1 \tag{1.16}$$

and

$$\overline{X}_d(\omega_a + k\omega_s) = \overline{X}_d(\omega_a) \tag{1.17}$$

After similar calculations for normalized frequency ω, it can be seen that the period is equal to 2π, that is,

$$\overline{X}(\omega) = \overline{X}(\omega + k2\pi) \tag{1.18}$$

A graphic interpretation of equation (1.18) is shown in Fig. 1.6.

Another peculiarity of the discrete signal spectrum is the behaviour of its phase–frequency components. If the signal is represented by a real function of time, then the spectrum values at the symmetrical points, relative to $\omega = k\pi$ are complex conjugates:

$$\overline{X}_d(2\pi - \omega) = \overline{X}_d(\omega)^* \tag{1.19}$$

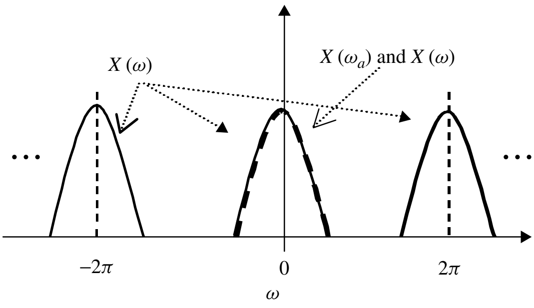

Figure 1.6 Spectrum of discrete signals

where $(\bullet)^*$ stands for a complex-conjugate value. Equation (1.19) directly follows from the simple formula

$$\overline{X}_d(2\pi - \omega) = \sum_{n=0}^{\infty} x(nT)\exp(\mathrm{j}\omega n) \cdot \exp(-\mathrm{j}2\pi n) = \sum_{n=0}^{\infty} x(nT)\exp(\mathrm{j}\omega n) \quad (1.20)$$

This peculiarity is an equivalent of the following relation between the amplitude and phase spectrum components

$$\begin{aligned} |\overline{X}_d(2\pi - \omega)| &= |\overline{X}_d(\omega)| \\ \theta_d(2\pi - \omega) &= -\theta_d(\omega) \end{aligned} \quad (1.21)$$

that correspond to the definition of the complex-conjugate function. Graphical interpretation of the equation is shown in Fig. 1.7.

It was shown earlier that the spectrum of the discrete signal is periodic. We can now determine the relations between the spectrum of an analog signal $\overline{X}(\omega_a)$ and the corresponding spectrum of a discrete signal $\overline{X}_d(\omega_a)$. In time domain, a discrete signal can be introduced via an appropriate analog signal as follows from equation (1.5)

$$x_d(t) = x(t) \cdot v(t) \quad (1.22)$$

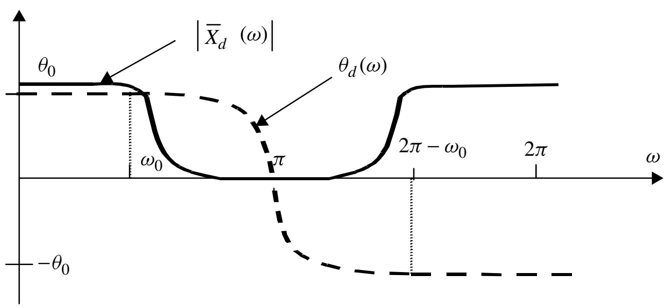

Figure 1.7 Amplitude and phase spectrum of a real discrete signal

It is known that a spectrum of the product of two functions is proportional to a convolution of these functions' spectrums [2]

$$\overline{X}_d(\omega_a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X}(\lambda)\cdot\overline{V}(\omega_a - \lambda)\,\mathrm{d}\lambda \tag{1.23}$$

where $\overline{X}(\lambda)$ is a spectrum of the initial analog signal $x(t)$ and $\overline{V}(\lambda)$ is a spectrum of the sampling signal $v(t)$. This sampling signal was specified earlier as a sequence of the δ-functions (1.6), the spectrum of which is

$$\overline{V}(\omega_a) = T\sum_{n=-\infty}^{\infty}\delta\left(\frac{\omega_a}{\omega_s} - n\right) \tag{1.24}$$

Consequently, combining (1.22) to (1.24) we obtain

$$\overline{X}_d(\omega_a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{X}(\lambda)T\sum_{n=-\infty}^{\infty}\delta\left(\frac{\omega_a-\lambda}{\omega_s} - n\right) \tag{1.25}$$

After integration and taking into account equation (1.8), we finally obtain the relation between $\overline{X}_a(\omega_a)$ and $\overline{X}_d(\omega_a)$:

$$\overline{X}_d(\omega_a) = \sum_{n=-\infty}^{\infty}\overline{X}(\omega_a - k\omega_s) \tag{1.26}$$

That is, the spectrum of the discrete signal $\overline{X}_d(\omega_a)$ is a sum of the spectrums $\overline{X}(\omega_a)$ of the initial analog signal shifted along the frequency with a period equal to the sampling frequency ω_s (Fig. 1.6). In other words, the spectrum of the discrete signal is periodic, and each component of this spectrum corresponds to the spectrum of the initial analog signal.

From a practical point of view, it is useful to consider the influence of the realistic sampling function waveform on the discrete signal spectrum. In this case, the sequence of δ functions should be replaced by the sequence of unit pulses with finite duration τ (Fig. 1.5b). This corresponds to the replacement of $v(t)$ on $v_\tau(t)$:

$$\begin{aligned}\overline{X}_d(\omega_a) &= \int_0^T x(t)v_\tau(t)\exp(-\mathrm{j}\omega_a t)\,\mathrm{d}t\\ &= \sum_{n=0}^{\infty}\int_{(nT-\tau/2)}^{nT+\tau/2} x(t)\exp(-\mathrm{j}\omega_a t)\,\mathrm{d}t\end{aligned} \tag{1.27}$$

Although τ is not an infinitely small value as in the δ-function, in practice it is still considerably less than the sampling period: $\tau \ll T$. Then, the integral in