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# POLARIZED LIGHT IN LIQUID CRYSTALS AND POLYMERS

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Toralf Scharf

Institute of Microtechnology  
University of Neuchâtel  
Switzerland



**WILEY-INTERSCIENCE**  
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LIQUID CRYSTALS AND  
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# PREFACE

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Liquid crystal displays are nowadays the standard for mobile/portable information displays. With the increasing interest in displays, the subject of liquid crystal optical devices has grown to become an exciting and expansive field of research and development. However, a fundamental difference between displays and more generalized optical elements is the resolution. An optical element needs accurate control of the shape of the wavefront over a certain area. To assure high quality, the lateral resolution has to be very high. To achieve that, pixilation is done by a diffractive approach and gradient index optics are used in relatively thick liquid crystal layers. While the first concept is the classical approach for information displays, discussion of gradient index optics with liquid crystals is rarely laid out. The goal of this work is to fill that gap and to enable effective analysis and design of high-resolution optical elements with liquid crystals and polymers.

This book presents the basic principles and provides a systematic treatment of light propagation in liquid crystals and polymers from ray tracing to Bragg reflection. It is intended to be a textbook for engineers and scientists as well as students. In addition, it is structured to serve as a reference book for optical design of photonic systems, including anisotropic materials as well. To best serve this range of requirements, the book covers three main subjects: theoretical analysis, practical characterization, and examples of concrete devices.

The theoretical section gives the basis for further analysis and repeats some well-known concepts for completeness. It adds some less common methods such as ray tracing and finite difference time domain simulations. The former method is used for the design of classical optical elements like prisms and lenses, while the latter is indispensable for the analysis and design of diffractive optics and texture analysis. Practical characterization of liquid crystal elements is often done with the polarization microscope. A chapter is dedicated to standard methods of observation. It is based on a minimum of theoretical understanding. Analysis of birefringent devices is only possible if the internal structure is analyzed. The local orientation of liquid crystals and polymers is given by their texture. A particular effort was made to discuss examples of how the texture translates into optical microscope images. The discussions help to analyze high-resolution optical devices and liquid crystal textures. It is known that technologies advance rapidly, but the underlying principles of operation remain. Therefore the discussion is focused in last parts on

example devices and systems to show how the theoretical concepts apply. Owing to the amount of published literature it is almost impossible to cover all of the details, therefore I have selected typical approaches.

The field of applied optics with anisotropic materials is interdisciplinary involving optics, materials science, and electronics. It is therefore difficult to cover all aspects adequately. I hope that this book is useful to understand effects, design new devices, and explain functional principles of the optics of liquid crystal and polymers.

Many of the results presented were produced in collaboration with several students. The author is grateful to Christian Bohley, Gerben Boer, Manuel Bouvier, and Sylvain Jaquet for their excellent work. The author also wishes to thank his colleagues at the University of Neuchâtel and especially at the Institute of Microtechnology for their help, discussions and continued support. Special thanks are given to Joachim Grupp, Naci Basturk, Rolf Klappert and Norbert Koptsis from ASULAB S.A. in Marin, and to Sung-Gook Park, Celestino Padeste, Helmut Schiff, and Jens Gobrecht from PSI Villigen. Without their help and technological support, it would be impossible for me to provide such a variety of examples on liquid crystal technology. The author also wishes to thank Christian Bohley and Gerben Boer for proofreading parts of the manuscript and their comments. I am particularly indebted to Ralf Stannarius, for his patient reading of the manuscript and helpful suggestions and corrections.

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September 2006*

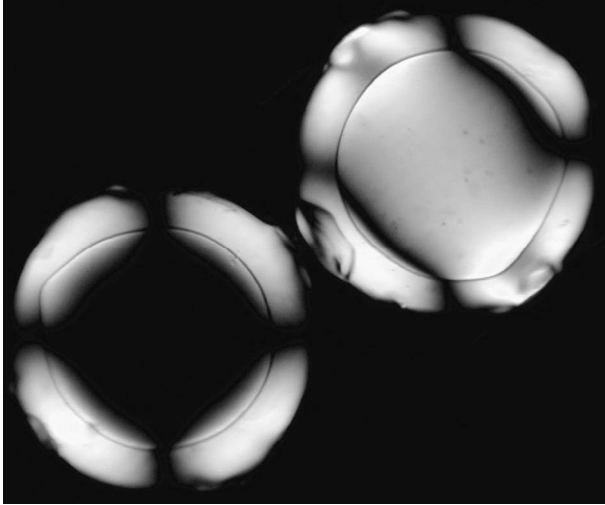
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# POLARIZED LIGHT

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## 1.1 INTRODUCTION

A normal beam of light in isotropic material consists of many individual waves, each vibrating in a direction perpendicular to its path. Measurable intensities therefore refer to a superposition of many millions of waves. Normally, the vibrations of each ray have different orientations with no favored direction. In some cases, however, all the waves in a beam vibrate in parallel planes in the same direction. Such light is said to be polarized, that is, to have a directional characteristic. More specifically, it is said to be linearly polarized, to distinguish it from circularly and elliptically polarized light (to be discussed later). Light from familiar sources such as a light bulb, the Sun, or a candle flame is unpolarized, but can easily become polarized as it interacts with matter. Such light is called natural light. Reflection, refraction, transmission, and scattering all can affect the state of polarization of light. The human eye cannot easily distinguish polarized from natural light. This is not true for all animals (Horváth, 2004). In fact, light from the sky is considerably polarized (Minnaert, 1954) as a result of scattering, and some animals, such as bees, are able to sense the polarization and use it as a directional aid. A major industrial use of polarized light is in photoelastic stress analysis (Föppl, 1972; Rohrbach, 1989; Dally, 1991). Models of mechanical parts are made of a transparent plastic, which becomes birefringent when stressed. Normal forces are applied to the model, which is then examined between polarizers. Between crossed polarizers, unstressed regions remain dark; regions under stress change the polarization of light so that light can be transmitted. Figure 1.1 shows an example of such stressed plastic parts



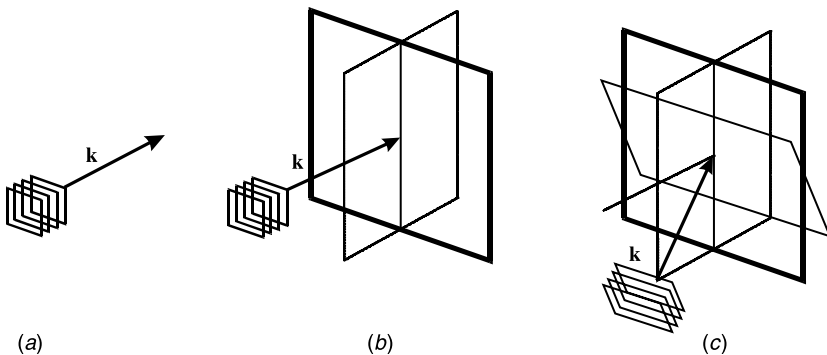
**Figure 1.1** Plastic disks of 1 mm diameter arranged between crossed polarizers. The disks are microfabricated by photolithography in a photo-polymerizable isotropic material (SU8). The polarizer and analyzer are horizontally and vertically oriented respectively. The left disk is not stressed, but the right is pressed together from top left and bottom right. The stress produces birefringence that creates interference colors.

between crossed polarizers. Disks of 1 mm diameter are shown. One is compressed and shows characteristic interference fringes due to stress. When white light is used, each wavelength is affected differently; the result is a highly colored contour map showing the magnitude and direction of the stresses. In Figure 1.1, the points where mechanical force is applied become visible as bright spots on the top-left and bottom-right positions. Another way to influence the polarization of light is optical activity. Many compounds are optically active; that is, they have the ability to rotate the plane of plane-polarized light. This property can be a molecular property and may be used to measure the concentrations of such compounds in solutions. More examples can be found in work by Pye (2001), Shurcliff (1962) and Minnaert (1954). Today, with advanced methods for measuring light fields now being available, polarized light still attracts a lot of attention. Recently, a discussion on singularities of light fields has led to a more intense discussion on polarization states and their propagations (Berry, 1999, 2003).

## 1.2 CONCEPT OF LIGHT POLARIZATION

Polarization is a property that is common to all types of vector waves. Electromagnetic waves also possess this property (Weller, 1979). For all types of vector waves, polarization refers to the behavior with time of one of the field vectors appropriate to that wave, observed at a fixed point in space. Light waves are electromagnetic in nature and require four basic field vectors for their complete description: the

electrical field strength  $\mathbf{E}$ , the electric displacement density  $\mathbf{D}$ , the magnetic field strength  $\mathbf{H}$ , and the magnetic flux density  $\mathbf{B}$ . Of these four vectors the electric field strength  $\mathbf{E}$  is chosen to define the state of polarization of light waves (Born, 1993). This choice is based on the fact that, when light interacts with matter, the force exerted on the electrons by the electric field of the light waves is much greater than the force exerted on these electrons by the magnetic field of the wave. In general, once the polarization of  $\mathbf{E}$  has been determined, the polarization of the three remaining vectors  $\mathbf{D}$ ,  $\mathbf{H}$ , and  $\mathbf{B}$  can be found, because the field vectors are interrelated by Maxwell's field equations and the associated constitutive (material) relations. In the following we will focus our attention on the propagation of light as defined by the behavior of the electric field vector  $\mathbf{E}(\mathbf{r}, t)$  observed at a fixed point in space,  $\mathbf{r}$ , and at time,  $t$ . In general the following statement holds: The change of polarization properties of light is initiated by a symmetry break while light propagates in a certain direction. This symmetry break can be made simply by the geometry of obstacles in the propagation path, by high dimensional order, or by anisotropy on a molecular or atomic level. Imagine a wave traveling in space (vacuum) in a direction described by a wave vector  $\mathbf{k}$  as shown in Figure 1.2. If such a wave hits a surface of an isotropic body of different refractive index at normal incidence, the symmetry with respect to the propagation direction is preserved and the state of polarization is not changed. The situation becomes different when the incidence is no longer normal to the surface. Now, a projection of the different vector components of the electromagnetic field has to be performed. This leads to different equations for reflection and transmission coefficients and finally to the Fresnel equations (see Chapter 3 for details), which allow the calculation of the change of the polarization state. Examples for highly ordered systems with anisotropy are crystals, liquid crystalline phases, and ordered polymers. Here, even for normal incidence, the anisotropy of the material can lead to a different interaction



**Figure 1.2** A plane wave traveling in direction  $\mathbf{k}$ . (a) The planes of constant phases (wavefronts) are indicated with the small layers. If the wave hits an interface normal as in (b), the symmetry of the problem is maintained. For isotropic materials no change of the polarization state of light is expected. If the light train has an oblique incidence as in (c) a change will be observed that leads to different reflection and transmission properties for differently polarized light trains.

of the vector components of the incident light: Projection of the electric field vectors on the symmetry axes of the system is needed to describe light propagation correctly. Matrix methods are convenient in this case to describe the change of polarization, for example the Jones matrix formalism (Chapter 4). Nanostructured materials with structures smaller than the wavelengths of light also fall into this category. An example is zero-order gratings with their particular polarization properties (Herzig, 1997). Light scattering also leads to polarization effects. For example, according to Rayleigh's classical scattering laws, any initially unpolarized beam becomes polarized when scattered. For this reason, the diffuse scattered light from the Sun in the atmosphere is partially polarized (Minnaert, 1954). On the molecular level, optical activity is an exciting case of interaction of polarized light with anisotropic molecules. Such a phenomenon is observed in sugar in solutions. The description is made by using a particular reference frame that is based on special states of light polarization (circularly polarized light). Optical activity is found naturally in crystals and can also be induced with electrical and magnetic fields and through mechanical stress.

Polarized light interaction happens on every length scale and is therefore responsible for a multitude of different effects. A description is particular difficult if all kinds of mechanisms overlap.

### 1.3 DESCRIPTION OF THE STATE OF POLARIZATION

To describe a general radiation field, four parameters should be specified: intensity, degree of polarization, plane of polarization, and ellipticity of the radiation at each point and in any given direction. However, it would be difficult to include such diverse quantities as intensity, a ratio, an angle, and a pure number in any symmetrical way in formulating the equation of propagation. A proper parametric representation of polarized light is therefore a matter of some importance. The most convenient representation of polarized light uses a set of four parameters, introduced by Sir George Stokes in 1852. One standard book on polarization optics (Goldstein, 2003) is based on this formulation and offers a deep insight into the formalisms by giving examples. It seems advantageous to use a description of light polarization that is linked to a measurement scheme. That is the case for the Stokes formalism. Assume for a moment that one has tools to separate the linear polarized (of different directions) and circular polarized light (of different sense of polarizations) from an incident light beam. If we know the direction of propagation of the light we are able to determine the properties related to polarization. Four values have to be measured to identify the state of polarization (including the ellipticity), the direction of the ellipse, and the degree of polarization (Gerrard, 1994). The scheme introduced by Stokes is based on the measurement of intensities by using ideal polarization components. Measurement of the total intensity  $I$  is performed without any polarizing component. Next, three intensities have to be measured when passing through an ideal polarizer (100% transmission for linear polarized light, 0% for extinction) at  $0^\circ$ ,  $45^\circ$  and  $90^\circ$  orientation, respectively. The coordinate system is fixed with respect to the direction of propagation. The last measurement uses a



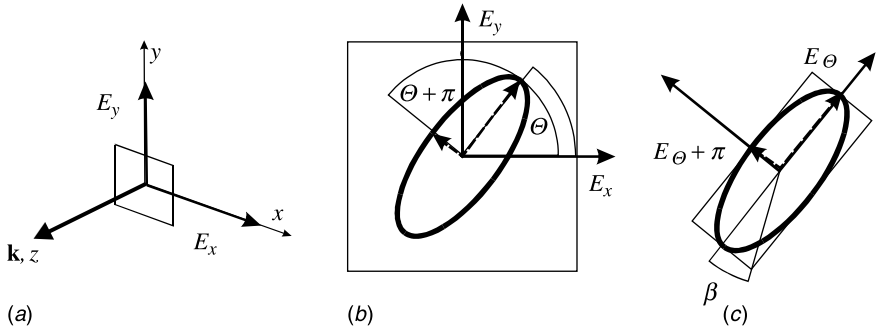
circular polarizer. All these measurements, together, allow the determination of the Stokes parameters  $S_0, S_1, S_2, S_3$ . The degree of polarization is given by comparing the total intensity with the sum of the ones measured with polarizers. The direction of the polarization ellipse can be found by analyzing the measurement with linear polarizers, carried out under different angles of the polarizer. The sense of rotation and the ellipticity is accessible when all measurements with polarizers are considered. To obtain this in a more quantitative manner, we start with a description of a transversal wave and the polarization states following the description in the work of Chandrasekhar (1960). We assume propagation in an isotropic material. The polarization state description is closely related to the propagation direction. In a ray model the propagation direction is easily defined as the vector of the ray direction. To have easy access to the main parameters of light polarization one assumes that the propagation direction is known and defines a coordinate system. Let  $z$  be the direction of propagation and the  $\mathbf{k}$  vector. Then the two components of the electrical field can be assigned with a phase and amplitude such that

$$E_x = E_{x0} \sin(\omega t + \varphi_x) \quad \text{and} \quad E_y = E_{y0} \sin(\omega t + \varphi_y), \quad (1.1)$$

where  $E_x$  and  $E_y$  are the components of the vibration along directions  $x$  and  $y$ , at right angles to each other,  $\omega$  is the angular frequency of vibration, and  $E_{x0}, E_{y0}, \varphi_x,$  and  $\varphi_y$  are constants. Figure 1.3 presents the geometrical arrangement. The field components vary in time with a certain phase shift and describe an ellipse. If the principal axes of the ellipse described by  $(E_x, E_y)$  are in directions making an angle  $\Theta$  and  $\Theta + \pi/2$  to the direction  $x$ , the equations representing the vibration take the simplified forms

$$E_\Theta = E_0 \cos \beta \sin \omega t \quad \text{and} \quad E_{\Theta+\pi/2} = E_0 \sin \beta \cos \omega t, \quad (1.2)$$

where  $\beta$  denotes an angle whose tangent is the ratio of the axis of the ellipse traced



**Figure 1.3** To a plane wave traveling in the  $z$ -direction with the wave vector  $\mathbf{k}$ , one can assign a plane with  $\mathbf{k}$  as the normal vector, which contains the field components of the electric field  $E_x$  and  $E_y$ .

by the end point of the electric vector as shown in Figure 1.3. We shall suppose that the numerical value of  $\beta$  lies between 0 and  $\pi/2$  and that the sign of  $\beta$  is positive or negative according to whether the polarization is right-handed or left-handed.  $E_0$  denotes a quantity proportional to the mean of the electric field vector, whose square is equal to the intensity  $\hat{I}$  of the beam:

$$\hat{I} = E_0^2 = E_{x0}^2 + E_{y0}^2 = \hat{I}_x + \hat{I}_y. \quad (1.3)$$

Note that the intensity  $\hat{I}$  introduced here is not the intensity measured by a detector. A detector measures time averaged values of the electric fields that is half of  $\hat{I}$  because of the time average of the trigonometric functions in Equation (1.1).

The formulas connecting the representations of polarized light with four intensities  $S_0, S_1, S_2,$  and  $S_3$  are important and can be combined in the following manner. The electrical field components of the field amplitude  $E_\Theta$  and  $E_{\Theta+\pi/2}$  in the  $x$ - and  $y$ -directions are summed (see Fig. 1.3). Starting from the representation in Equation (1.2) we obtain for the vibrations in the  $x$ - and  $y$ -direction the expressions

$$E_x = E_0(\cos \beta \cos \Theta \sin \omega t - \sin \beta \sin \Theta \cos \omega t) \quad (1.4a)$$

$$E_y = E_0(\cos \beta \sin \Theta \sin \omega t + \sin \beta \cos \Theta \cos \omega t). \quad (1.4b)$$

These equations can be reduced to the form of Equation (1.1) by letting

$$E_{x0} = E_0 \sqrt{\cos^2 \beta \cos^2 \Theta + \sin^2 \beta \sin^2 \Theta} \quad (1.5a)$$

$$E_{y0} = E_0 \sqrt{\cos^2 \beta \sin^2 \Theta + \sin^2 \beta \cos^2 \Theta} \quad (1.5b)$$

and

$$\tan \varphi_x = \tan \beta \tan \Theta \quad \text{and} \quad \tan \varphi_y = -\tan \beta / \tan \Theta. \quad (1.6)$$

The intensities  $\hat{I}_x$  and  $\hat{I}_y$  in the directions  $x$  and  $y$  are therefore given by

$$\hat{I}_x = E_{x0}^2 = I(\cos^2 \beta \cos^2 \Theta + \sin^2 \beta \sin^2 \Theta) \quad (1.7a)$$

$$\hat{I}_y = E_{y0}^2 = I(\cos^2 \beta \sin^2 \Theta + \sin^2 \beta \cos^2 \Theta). \quad (1.7b)$$

Further, according to Equations (1.5) and (1.6) we may readily verify that

$$2E_{x0}E_{y0} \cos(\varphi_x - \varphi_y) = 2E_0^2(\cos^2 \beta - \sin^2 \beta) \cos \Theta \sin \Theta = \hat{I} \cos 2\beta \sin 2\Theta. \quad (1.8)$$

Similarly,

$$2E_{x0}E_{y0} \sin(\varphi_x - \varphi_y) = \hat{I} \sin 2\beta. \quad (1.9)$$

From the foregoing equations (1.3–1.7) it follows that whenever the regular vibrations representing an elliptically polarized beam can be expressed in the form of Equation (1.1), we can at once write the relations

$$S_0 = \hat{I} = E_{x0}^2 + E_{y0}^2 = \hat{I}_x + \hat{I}_y \quad (1.10a)$$

$$S_1 = E_{x0}^2 - E_{y0}^2 = I \cos 2\beta \cos 2\Theta = \hat{I}_x - \hat{I}_y \quad (1.10b)$$

$$S_2 = 2E_{x0}E_{y0} \cos(\varphi_x - \varphi_y) = \hat{I} \cos 2\beta \sin 2\Theta = (\hat{I}_x - \hat{I}_y) \tan 2\Theta \quad (1.10c)$$

$$S_3 = 2E_{x0}E_{y0} \sin(\varphi_x - \varphi_y) = \hat{I} \sin 2\beta = (\hat{I}_x - \hat{I}_y) \frac{\tan 2\beta}{\cos 2\Theta}. \quad (1.10d)$$

These are the Stokes parameters representing an elliptically polarized beam. We observe that among the quantities  $S_0$ ,  $S_1$ ,  $S_2$ , and  $S_3$  defined as in Equations (1.10) there exists the relation

$$S_0^2 = S_1^2 + S_2^2 + S_3^2. \quad (1.11)$$

Further, the plane of polarization and the ellipticity follow from the equations

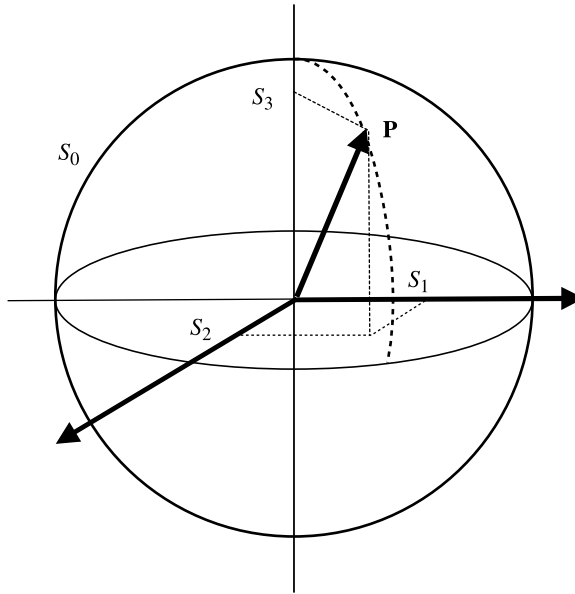
$$\tan 2\Theta = \frac{S_2}{S_1} \quad \text{and} \quad \sin 2\beta = \frac{S_3}{\sqrt{S_1^2 + S_2^2 + S_3^2}}. \quad (1.12)$$

For the representation given in Equation (1.1) one considers the amplitude and the phases to be constant. This is not realistic. The high angular frequency  $\omega$  of the electromagnetic oscillation of light allows us to suppose that the phases and amplitudes may be constant for millions of vibrations and yet changes irregularly millions of times a second. In an polarized light beam these irregular variations must be such that the ratio of amplitudes and the difference of phases are absolute constant. One is able to measure the mean intensity in any direction in the transverse plane. The apparent intensities  $\hat{I}_x$  and  $\hat{I}_y$  will be given by their mean values of electric field components  $E_x$  and  $E_y$ . Equations (1.10) can be rewritten by using mean intensities and allows to apply the formalism to measurements.

## 1.4 THE STOKES CONCEPT

Using Equations (1.10), the so-called Stokes column  $\mathbf{S}$  for completely polarized light is defined as

$$\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{x0}^2 + E_{y0}^2 \\ E_{x0}^2 - E_{y0}^2 \\ 2E_{x0}E_{y0} \cos(\varphi_x - \varphi_y) \\ 2E_{x0}E_{y0} \sin(\varphi_x - \varphi_y) \end{pmatrix}. \quad (1.13)$$



**Figure 1.4** Poincaré sphere representation of the polarization states of a monochromatic wave. In most definitions the sphere has a radius of 1. One can introduce the parameter  $S_0$  as the sphere radius to consider partially polarized light or different intensities.

The four elements of  $\mathbf{S}$  are directly obtained from intensity measurements and are related by the expression

$$S_0^2 = S_1^2 + S_2^2 + S_3^2, \quad (1.14)$$

which was shown above.  $S_0$  is the intensity of the beam,  $S_1$ ,  $S_2$ , and  $S_3$  can have any real value between  $-S_0$  and  $+S_0$ . Equation (1.14) allows a representation of the polarization states on the surface of a sphere, which is called a Poincaré sphere as shown in Figure 1.4. Here,  $S_1$ ,  $S_2$ , and  $S_3$  may be regarded as Cartesian coordinates of a point  $P$  on a sphere of radius  $S_0$ . Every possible state of polarization of a monochromatic plane wave corresponds to a point on the Poincaré sphere and vice versa. The right-handed circular polarization is represented by the north pole, the left-handed polarization by the south pole, the linear polarizations by points in the equatorial plane, and the elliptical states by the points between the poles and the equatorial plane. The right-handed polarization points lie above the equatorial plane and the left-handed lie below. According to a description of the polarization by complex numbers, the ensemble of the polarization states is mapped onto the complex plane; here it is mapped onto the surface of the Poincaré sphere and it exists as a unique projection between them.

Stokes columns can also describe partially polarized light. This can be thought of as combinations of several mutually incoherent beams of different polarizations. The combination can be obtained by addition of the intensities, which are represented by the individual elements of the Stokes column. These are directly related to the

so-called coherency matrix (Goodman, 2000). For partially polarized light the condition of Equation (1.14) will no longer be fulfilled. A measure for the validity of this condition can be the parameter  $p$ , given by

$$p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}, \quad (1.15)$$

which is called the degree of polarization. The degree of polarization  $p$  is equal to 1 for fully polarized light (mathematically, this is valid for all Stokes columns that are transformed directly from Maxwell columns) and equal to 0 for nonpolarized (natural) light. Stokes columns with a degree of polarization between 0 and 1 represent partially polarized light beams. Stokes vectors of partially polarized light beams can also be represented with the help of the Poincaré sphere. One can establish a matrices formalism that relates the Stokes vector of a light beam leaving an optical device with the Stokes vectors of the input beam. This matrix is called a Mueller matrix after its inventor. It is a  $4 \times 4$  matrix with real elements. The Stokes vectors  $\mathbf{S}$  are then transformed by

$$\mathbf{S}' = \begin{pmatrix} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}. \quad (1.16)$$

Not every real  $4 \times 4$  matrix can be a Mueller matrix  $\mathbf{M}$ . There exist conditions for testing the consistence of a matrix to be a Mueller matrix (Brosseau, 1998). These conditions can be formulated as inequalities. Mueller matrices can also be specified for nondepolarizing devices and then are called Mueller–Jones matrices.

## 1.5 THE JONES CONCEPT

In order to characterize the propagation of plane waves of light and the effect of anisotropic optical devices, it is convenient to use matrix methods. Several matrix methods have been developed (Jones, 1941, 1942, 1947, 1948, 1956; McMaster, 1961; Born, 1993). Under the condition of completely polarized beams a  $2 \times 2$  matrix method is sufficient for the description of the state of polarization. The light is represented only by its electric field and we examine a monochromatic plane wave of angular frequency  $\omega$ , which propagates in direction  $z$ , perpendicularly to the  $x$ – $y$ -plane. Then, the amplitude and phase behavior of the plane wave is determined by the two electric field vectors

$$E_x = E_{x0} e^{i(\omega t + \varphi_x)} \quad \text{and} \quad E_y = E_{y0} e^{i(\omega t + \varphi_y)}, \quad (1.17)$$

from which we obtain

$$\text{Re}(E_x) = E_{x0} \cos(\omega t + \varphi_x) \quad \text{and} \quad \text{Re}(E_y) = E_{y0} \cos(\omega t + \varphi_y). \quad (1.18)$$

$E_{x0}$  and  $E_{y0}$  are the amplitudes of the two electric fields,  $\varphi_x$  and  $\varphi_y$  their phases with a phase difference ( $\varphi_y - \varphi_x$ ). Note that here the definition of the phase difference is consistent with that in Section 1.3, although the absolute phases are different.

The column with the elements  $E_x$  and  $E_y$  is called the Maxwell column, and the corresponding time-independent vector

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{x0}e^{i\varphi_x} \\ E_{y0}e^{i\varphi_y} \end{pmatrix} \quad (1.19)$$

is called the Jones vector. The polarization of the plane wave described by Equation (1.19) can be defined by using a complex number notation. The intensity of  $\mathbf{E}$  is  $\mathbf{E}\mathbf{E}^*$  (\* means the complex conjugate). The vector in Equation (1.19) has the same polarization and intensity as

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{x0} \\ E_{y0}e^{i(\varphi_y - \varphi_x)} \end{pmatrix}. \quad (1.20)$$

There is a phase shift  $\varphi_x$  relative to Equation (1.19).

A polarization-dependent device can be characterized by a  $2 \times 2$  matrix  $\mathbf{J}$  that connects the incoming and outgoing waves in the Jones vector description by

$$\mathbf{E}_{\text{out}} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \quad (1.21)$$

Note that the elements of  $\mathbf{J}$  are in general complex. The Jones matrix describes the linear transformation of the Jones vector of a plane wave by reflection or transmission. The Jones matrix of a device that is composed of several devices in series can be obtained as the product of the Jones matrices of the individual devices. Jones matrices can be measured with special setups by analyzing the outgoing intensities for incoming beams of different polarizations.

For nondepolarizing devices there exists a mathematical correspondence between the Mueller–Jones and Jones matrices, namely

$$\mathbf{M} = \mathbf{A}(\mathbf{J} \otimes \mathbf{J}^*)\mathbf{A}^{-1}, \quad (1.22)$$

where  $\mathbf{J}$  is the corresponding Jones matrix,  $\mathbf{M}$  the corresponding Mueller matrix, and  $\mathbf{A}$  is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix}, \quad (1.23)$$

where  $\otimes$  designates the so-called Kronecker product (or outer product), which transforms two  $2 \times 2$  matrices into a  $4 \times 4$  matrix by multiplication of sub-

matrices component for component. The form of  $\mathbf{M}$  in terms of the Jones matrix elements is

$$\begin{pmatrix} \frac{1}{2}(|J_{11}|^2 + |J_{12}|^2 + |J_{21}|^2 + |J_{22}|^2) & \frac{1}{2}(|J_{11}|^2 - |J_{12}|^2 + |J_{21}|^2 - |J_{22}|^2) \\ \frac{1}{2}(|J_{11}|^2 + |J_{12}|^2 - |J_{21}|^2 - |J_{22}|^2) & \frac{1}{2}(|J_{11}|^2 - |J_{12}|^2 - |J_{21}|^2 + |J_{22}|^2) \\ \operatorname{Re}(J_{11}^* J_{21} + J_{12}^* J_{22}) & \operatorname{Re}(J_{11}^* J_{21} - J_{12}^* J_{22}) \\ \operatorname{Im}(J_{11}^* J_{21} + J_{12}^* J_{22}) & \operatorname{Im}(J_{11}^* J_{21} - J_{12}^* J_{22}) \\ \operatorname{Re}(J_{11}^* J_{12} + J_{21}^* J_{22}) & -\operatorname{Im}(J_{11}^* J_{12} + J_{21}^* J_{22}) \\ \operatorname{Re}(J_{11}^* J_{12} - J_{21}^* J_{22}) & -\operatorname{Im}(J_{11}^* J_{12} - J_{21}^* J_{22}) \\ \operatorname{Re}(J_{11}^* J_{22} + J_{12}^* J_{21}) & -\operatorname{Im}(J_{11}^* J_{22} - J_{12}^* J_{21}) \\ \operatorname{Im}(J_{11}^* J_{22} + J_{12}^* J_{21}) & \operatorname{Re}(J_{11}^* J_{22} - J_{12}^* J_{21}) \end{pmatrix}. \quad (1.24)$$

Mueller matrices of optical devices can be determined by Stokes vector measurements for different polarizations of the incident light, also for partially polarized light. Examples for Jones and Mueller matrices of particular devices and their measurement schemes can be found in the works of Gerrard (1994) and Brosseau (1998).

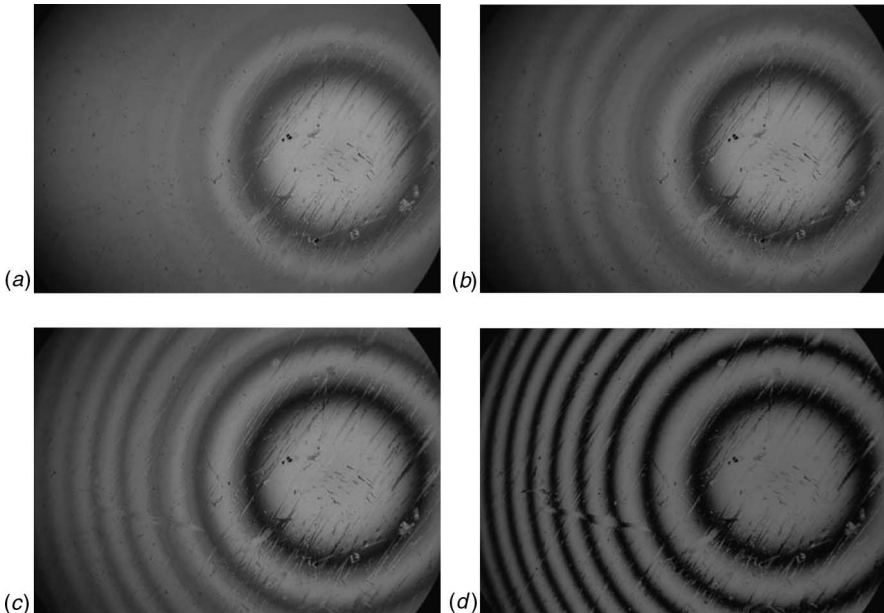
## 1.6 COHERENCE AND POLARIZED LIGHT

One particular beauty of experiments with polarized light is the appearing of colors appearing between crossed polarizers. This is due to interference effects in polarized light. However, interference is only possible under certain circumstances. This chapter will explain conditions when interference occurs and what kind of description is adequate for handling such problems. Three topics will be discussed: the effect of the spectra on interference, the effect of the quality of the source on interference, and the conditions for interference with polarized light.

### 1.6.1 Spectra and Coherence

A great majority of optical sources emit light by means of spontaneous emission from a collection of excited atoms or molecules, as is the case for the Sun, incandescent bulbs, and gas discharge lamps, for example. Such radiation, consisting of a large number of independent contributions, is referred to as thermal light. This kind of light must ultimately be treated as a random process (Goodman, 2000). In the preceding chapter it was assumed that light is deterministic or “coherent.” An exception was the description of partially polarized light with Stokes columns. An example of completely coherent light is a monochromatic wave with a description like  $\mathbf{U} = \operatorname{Re}(\mathbf{U}_0 e^{i\omega t})$ , for which the complex amplitude  $\mathbf{U}_0$  is a deterministic complex valued function, for example,  $\mathbf{U}_0 = \mathbf{E}_0 e^{i\mathbf{k}\mathbf{r}}$  in the case of a plane wave. The dependence of the wave function on time and space is perfectly periodic and

therefore predictable. If the light train is composed of several frequencies forming a continuous spectrum, the situation is more complicated. As a result, the temporal coherence of light influences the appearance of the interference effects. To quantify the influence one considers the spectrum of light as confined to a narrow band centered at a central frequency  $\nu_0$  or wavelength  $\lambda_0$ . The spectral width, or linewidth, of light is the width of  $\Delta\nu$  of the spectral density  $S(\nu)$  (Saleh, 1991). The coherence time is introduced as a parameter of characterization. A light source of broad spectrum has a short coherence time, whereas a light source with narrow linewidth has a long coherence time. For a given coherence time  $\tau_c$ , the distance that light travels in this time is  $l_c = c \tau_c$ . An experiment that produces interference effects with different path differences around  $l_c$  provides information about the coherence of light. The coherence properties of light can be demonstrated by an experiment where a birefringent material of varying thickness is put between crossed polarizers. A possible experimental setup is as follows. A convex lens and a microscope slide are treated with a polyimide and rubbed to form a uniform parallel aligned liquid crystal cell of variable thickness. This cell is filled with a highly birefringent liquid crystal mixture, such as BL006 from Merck ( $\Delta n = 0.28$ , clearing temperature  $T_{NI} = 110^\circ\text{C}$ ). Such a sample is observed between crossed polarizers where the rubbing direction is set to  $45^\circ$  with respect to the analyzer. Figure 1.5



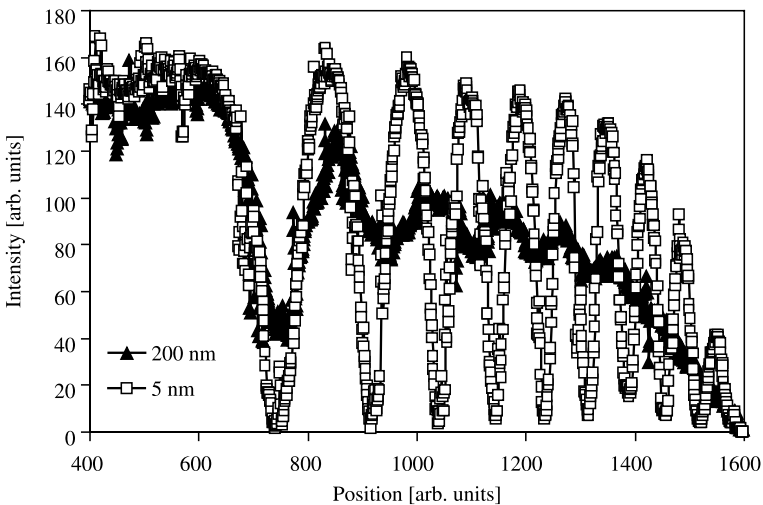
**Figure 1.5** Interference effects of a varying thickness liquid crystal cell observed between crossed polarizers. The thickness increases from the center of the round fringe at the right to the left. The width of the spectra of illumination is changed. It can be observed that the contrast of the interference fringes varies for different types of spectra. The smaller the bandwidths of the spectra the higher the contrast. (a) White light,  $\Delta\lambda = 400$  nm at  $\lambda_0 = 550$  nm; (b)  $\Delta\lambda = 200$  nm at  $\lambda_0 = 550$  nm; (c)  $\Delta\lambda = 100$  nm at  $\lambda_0 = 550$  nm; (d)  $\Delta\lambda = 5$  nm at  $\lambda_0 = 550$  nm.



shows the interference fringe system that can be seen for different illumination conditions.

The coherence of a light source can be increased by using optical filters to reduce its spectral width. To illustrate this effect, Figure 1.5 shows the result of increased coherence on the interference fringes. For white light, only the central fringe appears in good contrast. If the spectrum is narrowed to 200 nm, the contrast increases remarkably and additional fringes become visible. Decreasing the spectral width to 100 nm and finally to 5 nm leads to pronounced fringes at the highest contrast. The visibility of fringes is a measure of the coherence of light. A plot of the intensity over position for two different spectra is shown in Figure 1.6. The contrast of fringes is highest for the bandwidth of 5 nm and much smaller for the larger spectrum of 200 nm bandwidth. The positions of the peaks of maximum intensity are not the same for the different illuminations.

In the measurements above we have used rectangle-shaped spectral functions. The spectral width is the difference in width of the smallest and highest frequencies in the spectra. But the shape of the spectral density can have a different form and a definition of the spectral width becomes necessary. There are several definitions of spectral widths. The most common is the full width at half maximum (FWHM) of the spectral density  $S(\nu)$ . The relation between the coherence time and spectral width depends on the spectral profile, as indicated in Table 1.1. A particular situation arises if the light has no continuous spectrum. In particular, in the case of discharge lamps with spectral lines, the definition of the coherence time needs special care. In general, it is possible to define an equivalent coherence time. In such an approach, actual coherence properties can be simulated by using weighted averages.



**Figure 1.6** Measured intensity for two different spectral widths: 200 nm and 5 nm. The mean intensity is the same. The fringe contrast is reduced for broadband illumination with the 200 nm bandwidth.

**Table 1.1 Relation between spectral width and coherence time. Saleh, Teich 1991.**

Spectral Density Shape	Spectral Width $\Delta\nu_{FWHM}$
Rectangular	$\frac{1}{\tau_c}$
Lorentzian	$\frac{1}{\pi\tau_c} \approx \frac{0.32}{\tau_c}$
Gaussian	$\frac{(2 \ln 2/\pi)^{1/2}}{\tau_c} \approx \frac{0.66}{\tau_c}$

### 1.6.2 Spatial Coherence

An additional important parameter, which determines the visibility of fringes in an interference experiment, is the spatial coherence. Light sources have in general a certain dimension. An area of coherent emission related to this is called the coherence area. The coherence area is an important parameter that characterizes random light and must be considered in relation to other dimensions of the optical system. For example, if the area of spatial coherence is greater than the size of the aperture through which light is transmitted, the light may be regarded as spatially coherent. Similarly, if the coherence area is much smaller than the resolution of the optical system, it has to be regarded as incoherent. Light radiated from an extended radiating hot surface has an area of coherence in the order of  $\lambda^2$ , where  $\lambda$  is the central wavelength. In most practical cases it must be regarded as incoherent. Interference occurs when coherent waves are superimposed. Those portions of the light field that produce interference have to be spatially coherent. The spatially coherent parts of light can be identified as spatial modes of the light field. The light field is considered as being composed of a number of such modes. These modes can have any shape. The shape depends on the boundary conditions imposed when light is generated or detected. Independent of the particular mode shape, there exists a relation between the cross-sectional area  $A_M$  and the solid angle  $\Omega_M$ . For a mode at a given position this relation is always given by

$$A_M \Omega_M = \lambda^2, \tag{1.25}$$

where  $\lambda$  is the wavelength of the light. Equation (1.25) links properties of a spatially coherent emitting area  $\lambda^2$  to parameters of a coherent light beam. As only light that is spatially coherent and belonging to the same spatial mode can interfere, the coherent power of the light is equal to the power  $P_{mode}$  in one spatial mode.  $P_{mode}$  is directly related to the radiance  $B$ ,

$$B = \frac{dP}{dA d\Omega}, \tag{1.26}$$

which is the power  $dP$  per unit area  $dA$  and unit solid angle  $d\Omega$ , through

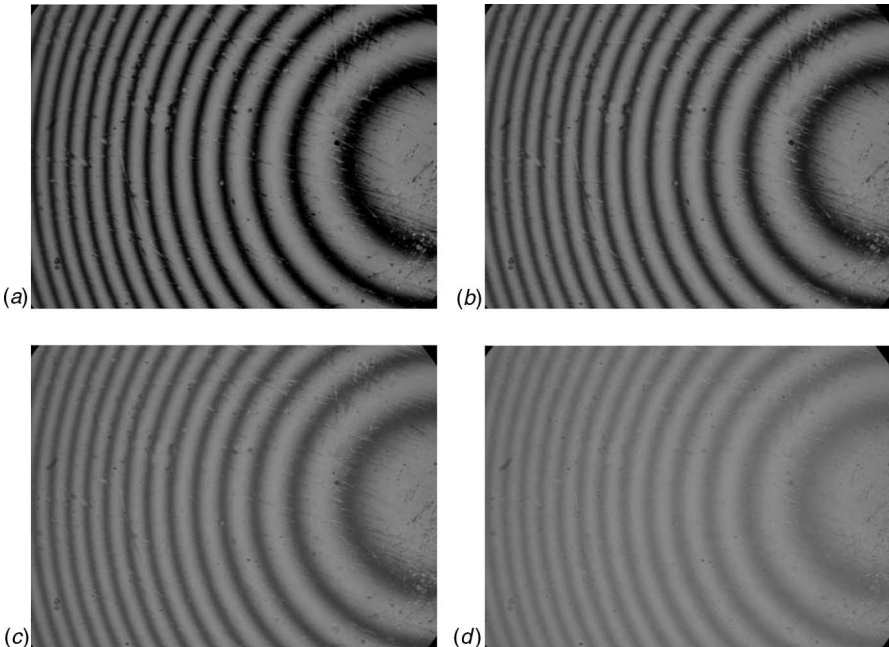
$$P_{\text{mode}} = B\Omega_M A_M = B\lambda^2. \quad (1.27)$$

The radiance  $B$ , as defined in Equation (1.26), measures the quality of light in terms of coherence. Liouville's theorem of optics states that the radiance  $B$  cannot be increased by passive optical elements such as lenses or mirrors. This means also that the coherent power of the power per mode  $P_{\text{mode}}$  cannot be increased by passive elements.

### 1.6.3 Coherence and Polarization

There are additional conditions to be satisfied in order to observe interference effects with polarized light. They are referred to as the rules of Fresnel and Arago (Goldstein, 2003):

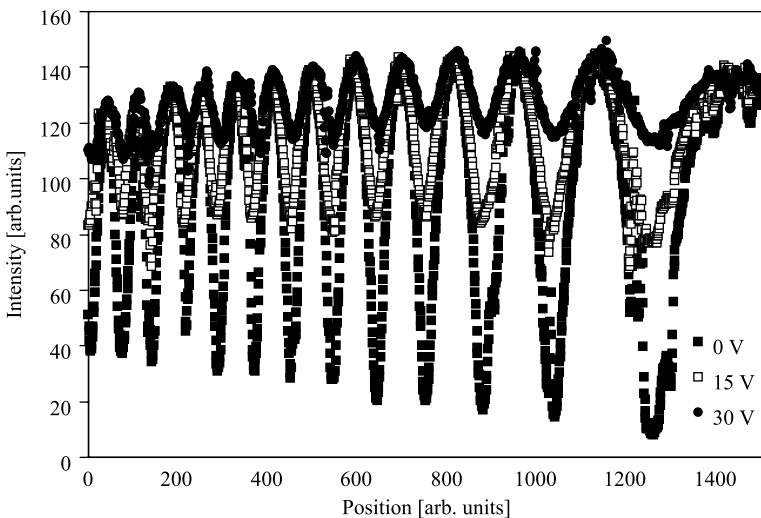
1. Two waves linearly polarized in the same plane can interfere.
2. Two waves linearly polarized with perpendicular polarizations cannot interfere.



**Figure 1.7** Interference fringes of a liquid crystal cell of varying thickness with planar alignment for different voltages: (a) 0 V; (b) 10 V; (c) 15 V; and (d) 30 V. The contrast of the fringes varies as a function of the degree of polarization of light. For ideal conditions, in 0 V, maximum contrast is found. Increasing the voltage reduces the degree of polarization of the incoming light and the contrast is reduced.

3. Two waves linearly polarized with perpendicular polarizations, if derived from the perpendicular components of unpolarized light and subsequently brought into the same plane, cannot interfere.
4. Two waves linearly polarized with perpendicular polarizations, if derived from the same linearly polarized wave and subsequently brought into the same plane, can interfere.

These conditions can be worked out with the Stokes formalism and it is interesting to see how this formalism applies (Goldstein, 2003). We will not treat the different cases here. We shall instead show the influence of the degree of polarization on interference with the example discussed above. Using liquid crystal display technology it is possible to build switchable polarizers with dichroic dyes diluted in liquid crystals. The performance, that is, the extinction ratio or pure transmission, of such a polarizer is not very good but still sufficient to perform the experiment. The dichroic liquid crystal cell in planar alignment with a positive dichroic dye absorbs electromagnetic energy for one polarization direction (Bahadur, 1990). As a result, the outgoing beam becomes polarized. If the absorption is not complete only a part of the concerned electrical field component is reduced in amplitude. This results in partial polarization. The degree of polarization can be changed by influencing the strength of absorption of the dichroic dye. With the dichroic liquid crystal cell this may be achieved by changing the orientation of the dichroic dye molecules in the liquid crystal host by applying a voltage (Bahadur, 1990). The result is an electrically switchable polarizer that allows the choice of degree of



**Figure 1.8** Intensity as a function of position for different voltages for the same measurement setup as described in the text. When the degree of polarization is changed, the contrast of fringes is varied but the positions of the maxima and minima remain the same. The contrast reduction from right to left is a result of scattering present in the system.

polarization. A microscope was equipped with such a polarizer to observe the interference fringes of a variable thickness cell. Figure 1.7 shows the case when the cell of variable thickness is illuminated with green light at 550 nm (spectral width  $\Delta\lambda = 5$  nm). For increasing voltage, the degree of polarization is reduced. In Figure 1.7 photographs of the interference fringes are shown for different voltages. Increasing the voltage leads to a decrease of contrast for the interference fringes. Only the polarized part can produce interference fringe contrast. The higher the voltage is the viewer the intensity of the polarized light becomes. The fringe contrast decreases but the overall intensity is maintained. To prove this, a plot of intensity as a function of position is given in Figure 1.8.

The positions of the maxima and minima stay the same as seen in Figure 1.8; only the contrast is changed. In comparison to Figure 1.6, the maximum intensity is always the same and the minimum changes with voltage.

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# ELECTROMAGNETIC WAVES IN ANISOTROPIC MATERIALS

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## 2.1 INTRODUCTION

The Maxwell equations are the foundations of modern electromagnetic theory. Maxwell's original formulation comprised 20 equations in 20 variables. The modern mathematical formulation of Maxwell's equations is credited to Oliver Heaviside and Willard Gibbs, who reformulated Maxwell's original system of equations and introduced the vector calculus. The vector notation produced a symmetric mathematical representation. This highly symmetrical formulation inspired later developments in fundamental physics. The electromagnetic field equations have an intimate link with special relativity; the magnetic field equations can be derived from consideration of the transformation of the electric field equations under relativistic transformations at low velocities.

## 2.2 ANALYTICAL BACKGROUND

Maxwell's equations describe all (classical) electromagnetic phenomena. With the electrical field vector  $\mathbf{E}$ , the magnetic field vector  $\mathbf{H}$ , the electric displacement vector  $\mathbf{D}$ , and the magnetic induction vector  $\mathbf{B}$  they can be expressed as (Born, 1993)

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \operatorname{div} \mathbf{B} &= 0, & \operatorname{div} \mathbf{D} &= \rho,\end{aligned}\tag{2.1}$$

where  $\mathbf{J}$  is the current and  $\rho$  is the electrical charge. The first equation is Faraday's law of induction, the second is Ampere's law as amended by Maxwell to include the displacement current  $\partial\mathbf{D}/\partial t$ , the third and fourth are Gauss's laws for the electric and magnetic fields. The displacement current term  $\partial\mathbf{D}/\partial t$  in Ampere's law is essential in predicting the existence of propagating electromagnetic waves. Equations (2.1) are in SI units. The quantities  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic field intensities and are measured in units of [V/m] and [A/m], respectively. The quantities  $\mathbf{D}$  and  $\mathbf{B}$  are the electric and magnetic flux densities, expressed in units of [C/m<sup>2</sup>] and [Weber/m<sup>2</sup>], or [Tesla].  $\mathbf{B}$  is also called the magnetic induction. The quantities  $\rho$  and  $\mathbf{J}$  are the volume charge density and electric current density (charge flux) of any external charges (i.e., not including any induced polarization charges and currents.) They are measured in units of [C/m<sup>3</sup>] and [A/m<sup>2</sup>]. The right-hand side of the equation  $\text{div } \mathbf{B} = 0$  is zero because there are no magnetic monopole charges.

The charge and current densities  $\rho$ ,  $\mathbf{J}$  may be thought of as the sources of the electromagnetic fields. For wave propagation problems, these densities are localized in space. The generated electric and magnetic fields are radiated away from these sources and can propagate to large distances compared to the wavelength. Away from the sources, that is, in source-free regions of space, Maxwell's equations take the simpler form:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \\ \text{div } \mathbf{B} &= 0, & \text{div } \mathbf{D} &= 0.\end{aligned}\quad (2.2)$$

The electric and magnetic flux densities  $\mathbf{D}$ ,  $\mathbf{B}$  are related to the field intensities  $\mathbf{E}$ ,  $\mathbf{H}$  via the so-called constitutive relations. The precise form of those depends on the material in which the fields exist. In vacuum, they take their simplest form:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad (2.3)$$

where  $\varepsilon_0$ ,  $\mu_0$  are the permittivity and permeability of vacuum, with numerical values

$$\varepsilon_0 = 8.854 \times 10^{-12} \frac{\text{As}}{\text{Vm}}, \quad \mu_0 = 4\pi \times 10^{-7} \frac{\text{Vs}}{\text{Am}}. \quad (2.4)$$

The units for  $\varepsilon_0$  and  $\mu_0$  are the units of the ratios  $\mathbf{D}/\mathbf{E}$  and  $\mathbf{B}/\mathbf{H}$ , that is, C/m<sup>2</sup> and V/m. From the two quantities  $\varepsilon_0$  and  $\mu_0$  one can define two other physical constants: the speed of light  $c_0$  and characteristic impedance of vacuum  $\eta_0$ :

$$c_0 = \sqrt{\frac{1}{\varepsilon_0 \mu_0}} = 2.9998 \times 10^8 \text{ m/s}, \quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \Omega. \quad (2.5)$$

The next simplest form of the constitutive relations is for simple dielectrics and for magnetic materials:

$$\mathbf{D} = \varepsilon_a \mathbf{E}, \quad \mathbf{B} = \mu_a \mathbf{H}. \quad (2.6)$$



These are typically valid at low frequencies. The permittivity  $\epsilon_a$  and permeability  $\mu_a$  are related to the electric and magnetic susceptibilities of the material as follows:

$$\epsilon_a = \epsilon_0(1 + \chi), \quad \mu_a = \mu_0(1 + \chi_m). \quad (2.7)$$

The susceptibilities  $\chi$  and  $\chi_m$  are measures of the electric and magnetic polarization properties of the material. For example, we have for the electric flux density

$$\mathbf{D} = \epsilon_a \mathbf{E} = \epsilon_0(1 + \chi)\mathbf{E} = \epsilon_0 \mathbf{E} + \chi \epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (2.8)$$

where the quantity  $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$  represents the dielectric polarization of the material, that is, the average electric dipole moment per unit volume. The speed of light in the material and the characteristic impedance are given by

$$c = \sqrt{\frac{1}{\epsilon_a \mu_a}}, \quad \eta = \sqrt{\frac{\mu_a}{\epsilon_a}}. \quad (2.9)$$

The relative permittivity and refractive index of the material are defined as

$$\epsilon = \frac{\epsilon_a}{\epsilon_0} = 1 + \chi, \quad n = \sqrt{\epsilon} \quad (2.10)$$

so that  $\epsilon = n^2$  and  $\epsilon \hat{\mathbf{a}} = \epsilon_0 \epsilon_r = \epsilon_0 n^2$ . Using the definition of Equation (2.10) and assuming a nonmagnetic material ( $\mu = \mu_0$ ), we may relate the speed of light and impedance of the material to the corresponding vacuum values:

$$c = \sqrt{\frac{1}{\epsilon_a \mu_a}} = \sqrt{\frac{1}{\epsilon_0 \epsilon \mu_0}} = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \frac{1}{n} = \frac{c_0}{n}. \quad (2.11)$$

Similarly, in a magnetic material, we have  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ , where  $\mathbf{M} = \chi_m \mathbf{H}$  is the magnetization, that is, the average magnetic moment per unit volume.

More generally, constitutive relations may be inhomogeneous, anisotropic, non-linear, frequency-dependent (dispersive), or all of the above. In inhomogeneous materials, the permittivity  $\epsilon$  depends on the location  $\mathbf{r}$  within the material:

$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \epsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}, t). \quad (2.12)$$

In anisotropic materials,  $\epsilon$  depends on direction and becomes a tensor. The constitutive relations may be written component-wise in matrix (or tensor) form as

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \quad (2.13)$$

It is important to note that for a lossless and nonoptically active medium,  $\varepsilon_{ij} = \varepsilon_{ji}$ . More complicated forms of constitutive relationships arise in chiral and gyrosopic media and are discussed elsewhere (Serdyukov et al., 2001).

The cross-product of the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{H}$  is often called the Poynting vector, named after its inventor John Henry Poynting. For an electromagnetic wave it points in the direction of energy flow and its magnitude is the power per unit area crossing a surface that is normal to it. It is given the symbol  $\mathbf{S}$  (in bold because it is a vector),

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu\mu_0} \mathbf{E} \times \mathbf{B}, \quad (2.14)$$

where  $\mathbf{E}$  is the electric field,  $\mathbf{H}$  and  $\mathbf{B}$  are the magnetic field and magnetic flux density respectively, and  $\mu$  is the permeability of the surrounding medium. For an electromagnetic wave propagating in free space  $\mu$  becomes equal to 1, the permeability of free space. As the electric and magnetic fields of an electromagnetic wave oscillate, the magnitude of the Poynting vector changes with time. The average of the magnitude over a long time  $T$  (longer than the period of the wave) is called the irradiance,  $I$ :

$$I = \langle S \rangle_T. \quad (2.15)$$

### 2.3 TIME HARMONIC FIELDS AND PLANE WAVES

Maxwell's equations simplify considerably in the case of harmonic time dependence. Through the Fourier transform, general solutions of Maxwell's equation can be built as linear combinations of single-frequency solutions:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{i\omega t} d\omega. \quad (2.16)$$

Thus, we assume that all fields have a time dependence  $e^{i\omega t}$ :

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) e^{i\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r}) e^{i\omega t}, \quad (2.17)$$

where the phasor amplitudes  $\mathbf{E}(\mathbf{r})$ ,  $\mathbf{H}(\mathbf{r})$  are complex-valued. Replacing time derivatives by  $\partial/\partial t \rightarrow i\omega$ , we may rewrite Equation (2.1) in the form

$$\begin{aligned} \nabla \times \mathbf{E} &= -i\omega \mathbf{B}, & \nabla \times \mathbf{H} &= \mathbf{J} + i\omega \mathbf{D} \\ \operatorname{div} \mathbf{B} &= 0, & \operatorname{div} \mathbf{D} &= \rho. \end{aligned} \quad (2.18)$$

In this book, we will consider solutions of Maxwell's equations in two different contexts:

1. Uniform plane waves propagating in dielectrics and birefringent media; and
2. Waves propagating in nonuniform dielectric and birefringent media.

It is therefore instructive to see the simplifications provided by the additional assumption of having plane waves. The plane wave propagating in direction  $\mathbf{r}$  is described by an amplitude vector  $\mathbf{E}$  and wave vector  $\mathbf{k}$ . For a monochromatic wave we assume the form

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \mathbf{E}e^{i(\omega t - \mathbf{k}\mathbf{r})} \\ \mathbf{H}(\mathbf{r}, t) &= \mathbf{H}e^{i(\omega t - \mathbf{k}\mathbf{r})}.\end{aligned}\quad (2.19)$$

The wave vector  $\mathbf{k} = (\omega/c)n\mathbf{s}$  contains  $\mathbf{s}$  as a unit vector. The unit vector  $\mathbf{s}$  is in the direction of propagation. The phase velocities  $c/n$ , or equivalently the refractive index  $n$ , are to be determined. Substituting  $\mathbf{E}$  and  $\mathbf{H}$  from Equation (2.19) into Maxwell's Equations (2.18) gives

$$\begin{aligned}\mathbf{k} \times \mathbf{E} &= \omega\mu\mu_0\mathbf{H} \\ \mathbf{k} \times \mathbf{H} &= -\omega\mathbf{D} = -\omega\boldsymbol{\varepsilon}\varepsilon_0\mathbf{E}.\end{aligned}\quad (2.20)$$

By eliminating the magnetic field  $\mathbf{H}$  in Equation (2.20) we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2\varepsilon_0\mu_0\boldsymbol{\varepsilon}\mu\mathbf{E} = 0. \quad (2.21)$$

Using the equality  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ , the first summand in Equation (2.21) takes the form

$$\begin{aligned}\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) &= \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - \mathbf{E}(\mathbf{k} \cdot \mathbf{k}) \\ &= \begin{pmatrix} -k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & -k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & -k_x^2 - k_y^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.\end{aligned}\quad (2.22)$$

This equation will now be used to solve for the eigenvectors of  $\mathbf{E}$  and the corresponding eigenvalues. In the principal coordinate system the dielectric tensor of nonchiral and nongyroscopic media is a diagonal tensor with three different entries. It can be written as

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}.\quad (2.23)$$

When used in Equation (2.21) and with Equation (2.22) this leads to

$$\begin{pmatrix} \omega^2\varepsilon_0\varepsilon_x\mu\mu_0 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\varepsilon_0\varepsilon_y\mu\mu_0 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\varepsilon_0\varepsilon_z\mu\mu_0 - k_x^2 - k_y^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \quad (2.24)$$

For nontrivial solutions to exist, the determinant of the matrix must vanish, hence

$$\det \begin{vmatrix} \omega^2 \varepsilon_0 \varepsilon_x \mu \mu_0 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \varepsilon_0 \varepsilon_y \mu \mu_0 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \varepsilon_0 \varepsilon_z \mu \mu_0 - k_x^2 - k_y^2 \end{vmatrix} = 0. \quad (2.25)$$

At a given frequency  $\omega$ , this equation represents a three-dimensional surface in  $\mathbf{k}$  space. This surface consist of two shells and is known as a normal surface. In general, four points are in common for these two shells. The two lines that go through the origin and these points are known as the optical axis.

## 2.4 MAXWELL'S EQUATIONS IN MATRIX REPRESENTATION

One interesting representation of Maxwell's equations that allows simplification for special cases is the matrix description (Azzam, 1987). To give an overview one starts with Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, \\ \operatorname{div} \mathbf{B} &= 0, & \operatorname{div} \mathbf{D} &= 0. \end{aligned} \quad (2.26)$$

Using the matrix description of derivations

$$\operatorname{rot} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} \quad \text{and} \quad \operatorname{div} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \quad (2.27)$$

one gets for Faraday's law of induction

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \operatorname{rot} \mathbf{E} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -\begin{pmatrix} \dot{B}_x \\ \dot{B}_y \\ \dot{B}_z \end{pmatrix}. \quad (2.28)$$

The set of Maxwell's equations can be expressed in a vector representation using all field components, electrical and magnetic ones, as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} \dot{D}_x \\ \dot{D}_y \\ \dot{D}_z \\ \dot{B}_x \\ \dot{B}_y \\ \dot{B}_z \end{pmatrix}. \quad (2.29)$$

We have still neglected the constitutive relations between  $\mathbf{E}$  and  $\mathbf{D}$  and  $\mathbf{H}$  and  $\mathbf{B}$ . Written in a more compact form

$$\begin{pmatrix} \mathbf{0} & \mathbf{rot} \\ -\mathbf{rot} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{D}} \\ \dot{\mathbf{B}} \end{pmatrix}. \quad (2.30)$$

The time derivative is indicated by the point (dot) above quantities. When using a  $6 \times 1$  column vector  $\mathbf{G}$ , whose elements are the Cartesian components of  $\mathbf{E}$ , followed by that of  $\mathbf{H}$ , and with  $\mathbf{C}$ , that is a  $6 \times 1$  column vector, whose elements are  $\mathbf{D}$ , followed by those of  $\mathbf{B}$ , the equation reads

$$\begin{pmatrix} \mathbf{0} & \mathbf{rot} \\ -\mathbf{rot} & \mathbf{0} \end{pmatrix} \mathbf{G} = \dot{\mathbf{C}} \quad \text{or alternatively} \quad \mathbf{O} \mathbf{G} = \dot{\mathbf{C}}. \quad (2.31)$$

$\mathbf{O}$  here is the  $6 \times 6$  matrix containing the derivations. In the absence of nonlinear optical effects and spatial dispersion, the constitutive relations  $\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$  and  $\mathbf{B} = \mu_0 \mu \mathbf{H}$  can be translated into  $\mathbf{C}$  and  $\mathbf{G}$ ; One finds

$$\mathbf{C} = \mathbf{M} \mathbf{G}, \quad (2.32)$$

where  $\mathbf{M}$  is the  $6 \times 6$  matrix.  $\mathbf{M}$  carries all the information about the anisotropic optical material properties. This matrix is called the optical matrix and can be partitioned as

$$\mathbf{M} = \begin{pmatrix} \varepsilon_0 \varepsilon & \boldsymbol{\rho} \\ \boldsymbol{\rho}' & \mu_0 \mu \end{pmatrix}, \quad (2.33)$$

with the following submatrices:

$$\begin{aligned}
 \boldsymbol{\varepsilon} &= \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}, & \boldsymbol{\mu} &= \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix}, \\
 \boldsymbol{\rho} &= \begin{pmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \rho_{zx} & \rho_{zy} & \rho_{zz} \end{pmatrix}, & \boldsymbol{\rho}' &= \begin{pmatrix} \rho'_{xx} & \rho'_{xy} & \rho'_{xz} \\ \rho'_{yx} & \rho'_{yy} & \rho'_{yz} \\ \rho'_{zx} & \rho'_{zy} & \rho'_{zz} \end{pmatrix}.
 \end{aligned} \tag{2.34}$$

The representation contains the dielectric permittivity tensor  $\boldsymbol{\varepsilon}$ , the magnetic permittivity tensor  $\boldsymbol{\mu}$ , and two optical rotation tensors  $\boldsymbol{\rho}$  and  $\boldsymbol{\rho}'$ . These follows the very compact form

$$\mathbf{OG} = \mathbf{M}\dot{\mathbf{G}}. \tag{2.35}$$

For time harmonic fields such as

$$\mathbf{G} = \mathbf{e}^{i\omega t} \boldsymbol{\Gamma}, \tag{2.36}$$

where  $\boldsymbol{\Gamma}$  is the spatial part of  $\mathbf{G}$ , Equation (2.35) becomes

$$\mathbf{O}\boldsymbol{\Gamma} = i\omega\mathbf{M}\boldsymbol{\Gamma} \quad \text{or equivalently} \quad (\mathbf{O} - i\omega\mathbf{M})\boldsymbol{\Gamma} = 0, \tag{2.37}$$

which is the spatial wave equation for frequency  $\omega$ . For nonchiral materials, when the optical activities tensors  $\boldsymbol{\rho}$  and  $\boldsymbol{\rho}'$  are zero, the equations can be simplified to

$$\begin{pmatrix} \mathbf{0} & \mathbf{rot} \\ -\mathbf{rot} & \mathbf{0} \end{pmatrix} \boldsymbol{\Gamma} = i\omega \begin{pmatrix} \varepsilon_0 \boldsymbol{\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mu_0 \boldsymbol{\mu} \end{pmatrix} \boldsymbol{\Gamma}. \tag{2.38}$$

That excludes only molecular-based optical activity or chirality, following from effective medium calculations. Consequently we find

$$\begin{pmatrix} i\omega\varepsilon_0\boldsymbol{\varepsilon} & -\mathbf{rot} \\ \mathbf{rot} & i\omega\mu_0\boldsymbol{\mu} \end{pmatrix} \boldsymbol{\Gamma} = 0. \tag{2.39}$$

This equation can now be transformed to a simpler form by rearranging the field components and taking into account the properties of the permeability's  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\mu}$ . Expanding Equation (2.39), under the assumption that the field components of  $\mathbf{E}$

and  $\mathbf{H}$  are now representing only spatial variation, we obtain

$$\begin{pmatrix} i\omega\varepsilon_0\varepsilon_{xx} & i\omega\varepsilon_0\varepsilon_{xy} & i\omega\varepsilon_0\varepsilon_{xz} & 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} \\ i\omega\varepsilon_0\varepsilon_{yx} & i\omega\varepsilon_0\varepsilon_{yy} & i\omega\varepsilon_0\varepsilon_{yz} & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ i\omega\varepsilon_0\varepsilon_{zx} & i\omega\varepsilon_0\varepsilon_{zy} & i\omega\varepsilon_0\varepsilon_{zz} & \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & i\omega\mu_0\mu_{xx} & i\omega\mu_0\mu_{xy} & i\omega\mu_0\mu_{xz} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & i\omega\mu_0\mu_{yx} & i\omega\mu_0\mu_{yy} & i\omega\mu_0\mu_{yz} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & i\omega\mu_0\mu_{zx} & i\omega\mu_0\mu_{zy} & i\omega\mu_0\mu_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} = 0. \quad (2.40)$$

In a further simplification we can assume that the magnetic permeability  $\mu$  is isotropic and can be written as

$$\boldsymbol{\mu} = \mu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.41)$$

If, further on, the dielectric permittivity tensor can be transformed to its diagonal form with a convenient rotation of the coordinate system,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \implies \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad (2.42)$$

one obtains for Equation (2.40)

$$\begin{pmatrix} i\omega\varepsilon_0\varepsilon_1 & 0 & 0 & 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} \\ 0 & i\omega\varepsilon_0\varepsilon_2 & 0 & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & i\omega\varepsilon_0\varepsilon_3 & \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & i\omega\mu_0\mu & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 & i\omega\mu_0\mu & 0 \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & i\omega\mu_0\mu \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} = 0. \quad (2.43)$$

Equation (2.43) contains 18 elements that are zero out of 36 elements. In principle, that would allow the rearrangement of the elements and separation of the zeros. Interchanging columns is possible only if the order of field vectors is changed too, and rows can be independently interchanged. If by suitable coordinate transformation a form with zero submatrices, such as

$$\begin{pmatrix} \mathbf{h} & 0 \\ 0 & \mathbf{g} \end{pmatrix} \Gamma' = 0, \quad (2.44)$$

can be found, the propagation of polarized light for two distinct polarizations can then be treated as independent. The state of polarization for these two eigenpolarizations of the system does not change when propagated through the medium. Note that we made several simplifications that made this possible:

- The material does not show optical activity; and
- The magnetic permeability is a scalar.

We will now discuss two particular examples where polarization separation is possible.

## 2.5 SEPARATION OF POLARIZATIONS FOR INHOMOGENEOUS PROBLEMS

The differential equation of Equation (2.40) holds for an inhomogeneous anisotropic material in three-dimensional space. Equation (2.40) can only be solved analytically in some special cases. Under certain circumstances it is possible to simplify the system of equations. For the three-dimensional problem the variations of the fields in any direction in space may be nonzero. Assuming isotropic material with a spatially varying dielectric constant  $\varepsilon(\mathbf{r})$  in Equation (2.43) we obtain

$$\begin{pmatrix} i\omega \varepsilon_0 \varepsilon(\mathbf{r}) & 0 & 0 & 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} \\ 0 & i\omega \varepsilon_0 \varepsilon(\mathbf{r}) & 0 & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & i\omega \varepsilon_0 \varepsilon(\mathbf{r}) & \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & i\omega \mu_0 \mu & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} & 0 & i\omega \mu_0 \mu & 0 \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & i\omega \mu_0 \mu \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} = \mathbf{0}. \quad (2.45)$$

The matrix on the left side in Equation (2.45) cannot be rearranged such that the  $6 \times 6$  matrix separates into nondiagonal  $3 \times 3$  submatrices, although there are 18