A Practical Guide to Reliable Finite Element Modelling

Alan Morris
Emeritus Professor of Computational Structural Analysis,
Cranfield University, UK

With contributions from

Ahmed Rahman
QinetiQ, UK

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To Lilian,
for endless patience.

Contents

5 Preliminary Review of Errors and Error Control 113

Preface

This book is not a standard finite element text that can be used to provide all the information required to obtain a well-grounded understanding of all aspects of the Finite Element Method. There are already many excellent books on this topic and a number of these are referenced in Chapter 1, Introduction. The question that the book attempts to answer is 'How can an error-controlled finite element analysis be performed?' It is tempting to think that with the development of comprehensive finite element packages there is no need to worry about errors and uncertainties. Sadly this is not the case.

The Sleipner oilfield within the Norwegian sector of the North Sea is one of the major sources of oil and gas for Europe. The Sleipner A platform is a concrete gravity base structure consisting of 24 cells that rest on the sea bed at a depth of 82 m with a total base area of 16,000 m^2. Four of these cells are elongated so that they reach above the surface of the sea and support a deck that weighs 57,000 tons and drilling equipment weighing 40,000 tons. On 23 August 1991, while being prepared for deck mating through a controlled ballasting operation in the Gandsfforden outside Stravanger, the first Sleipner A platform sprang a leak and sank. The crash caused a seismic event of 3.0 on the Richter scale, left a pile of debris at a depth of 220 m and an economic loss of £700 million. The cause of the crash was traced to an inaccurate finite element analysis that underestimated the shear stress in the cells by 47%, so that certain concrete walls were not thick enough. After the accident, a more careful finite element analysis was performed on the original Sleipner A platform which predicted a structural failure at 62 m matching well with the actual failure depth of 64 m. Had an effective quality system been in place that allowed the analysis team to control the errors and uncertainties in the analysis, this failure could have been avoided. It may be thought that because the event took place some time ago the current situation would be much better, but, to the author's personal knowledge, other serious analysis failures have taken place recently. These have not been publicised as legal action was taken but were resolved at the courtroom door following an agreed compensation package.

In order to avoid such distressing consequences an analyst needs to have both sufficient basic knowledge of the Finite Element Method and a procedure for systematically performing a finite element analysis. This book aims to satisfy both these needs by providing essential background knowledge and information and a sequential application process. The book draws on two sources. One is information from lectures developed at Cranfield University and given to postgraduate aeronautics students and industrial short courses given both at Cranfield and in-house at international aerospace companies in the UK and elsewhere. The second source is the research output from a major UK government-funded project, within the Safety Critical Systems initiative, entitled SAFESA^{TM}, under contract DTI/EPSRC project 9034. Five organisations were involved in the project: Cranfield University, Lloyds Register, W.S. Atkins, Nuclear Electric (now British Nuclear Group) and Assessment Services (now Siemens). The author is particularly grateful to a number of colleagues involved in the SAFESA project who worked for these companies: Dr Mike Fox, Dr John Maguire, Dr Nigel Knowles and Professor Rade Vignjvec. Through creative and innovative thinking these engineers came forward with concepts and ideas that have significantly influenced the contents of this book.

Although Chapter 9 draws on the output of the SAFESA project, the method presented therein is distinctive. Nevertheless, the reader may wish to take advantage of the earlier work and this can be done, at one level, by consulting references [1] and [2] which present a synopsis of the main SAFESA results. A fuller description can be found in the *SAFESA Technical Manual* that was issued to the technical community, at the conclusion of the project, by the Minister then in charge of the Department of Industry and Science, the Rt Hon. Michael Heseltine MP (now Lord Heseltine). The SAFESA project team subsequently gave the NAFEMS organisation permission to reprint the *Manual* and copies can be obtained from that organisation through its offices in East Kilbride, Glasgow, UK.

The companion website for the book is http://wiley.com/go/morrisfem

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1

Introduction

1.1 AIM OF THE BOOK

There are many excellent text books on finite element theory incorporating the development of specific types of finite elements and describing the associated solution processes. This book has a different purpose from these standard texts as it provides a practical guide for the reliable use of the Finite Element Method in supporting the design of complex structures. Within this broad framework it gives an introduction to the Finite Element Method and links it to the problems associated with creating an effective and relatively error-free finite element model for solving a real-world structural design problem. By error is meant the difference between the finite element analysis's predicted behaviour and response of a structure subjected to applied loads and that which occurs when the structure enters service where the in-service loads come into play.

In practical terms the book is intended to assist engineers and companies involved with finite element analysis on a regular basis to operate in a manner that:

1. Reduces the possibility that any type of error is introduced into a finite element analysis.
2. Ensures that analyses undertaken by an individual analyst or analysis team are performed to a consistent and reliable standard.
3. Provides documentary evidence of having adhered to a consistent error control process as a basis for a defence in legal proceedings

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A Practical Guide to Reliable Finite Element Modelling A. Morris

should a structural failure occur after a finite element analysed product has entered service.

Clearly one of the key aspects of the book is the provision of a methodology that allows a finite element analysis of a structure to be undertaken in such a manner that potential differences between the values for specific behaviour parameters obtained from the analysis and the measured values from operational use are identified and controlled. This requires that the analyst is not only able to identify the sources of error that may give rise to such differences, but also able to provide bounds on their maximum likely value. The targeted parameters should be selected by a process that clearly and explicitly defines the qualification criteria that, when satisfied, allow the structure to be constructed and enter service in a manner that renders it fit for purpose. In essence, the process is attempting to generate a procedure that places analysis as the primary route for the qualification of a structure. This creates a new environment in which testing is analysis controlled and is employed to support the analyst, providing information for the bounding or control of potential errors. In this situation, testing is a subservient activity because the analyst defines specific requirements for test data to compensate for identified deficiencies in the finite element analysis. If a test is now used in the proof of a structure, it is there simply to validate the analysis which has become the actual validating machine.

In attempting to satisfy the requirements listed above the book offers a basis for constructing a logical approach to finite element analysis. This is ambitious and it is not claimed that it provides a complete and totally comprehensive method for satisfying this requirement. Rather it provides a door through which the reader is invited to step and after crossing the threshold develop the ideas presented herein into a more comprehensive and authoritative method that is personal to an individual analyst or analysis team. In the case of an inexperienced or new finite element analyst, it provides a starting point. For an experienced analyst or a company that regularly undertakes finite element analyses, it should be taken as an input into what should be a regular review of their finite element qualification process.

In order to keep the length and complexity of the book under control the problem domain is restricted to linear static and linear dynamic structural analyses. Nevertheless, the broad approach adopted in the chapters devoted specifically to error control and treatment has general applicability.

Finally, it is worth noting that this book is not intended as a broad introduction to the use of finite element analysis in engineering design; this is covered by Adams and Askenazi [1]. Nor does it focus on the development of internal error bounds and the use of this type of bounding process in h- and p-type adaptive meshing codes. However, the use of such codes is touched on as they provide one component in a total error and uncertainty control methodology. Details of error estimation techniques based on internal and self-referencing procedures are covered in the excellent book by Szabó and Babuška cited as reference [2] and, in more detail, by reference [3].

1.2 FINITE ELEMENT TYPES – A BRIEF OVERVIEW

The underlying principle of the Finite Element Method is that a physical structure is modelled as an assemblage of individual elements as outlined in Chapters 2 and 3 but more fully in books addressing the mathematical fundamentals such as references [4], [5], [6]. All finite element models employ polynomial approximations to at least one of the main fields employed in describing the physical phenomena that are the focus of the analysis. In this book, attention is restricted to the analysis of loaded structures responding in a manner that can be modelled using elasticity theory. For this class of modelling problems there are three basic element types: displacement elements, equilibrium elements and hybrid elements. All commercially available finite element packages and systems employ displacement finite elements, many employ some hybrid elements and a few have equilibrium elements. Chapters 2 and 3, in outlining some of the fundamentals of the method, use displacement elements. However, most of the arguments advanced in this book apply equally to all three types.

A schematic of a displacement finite element is shown in Figure 1.1. The displacement on the interior of the element is approximated using relatively low-order polynomials. These polynomials must have a form that ensures the displacements at the edge or edge surfaces of the element can link up with adjacent elements in such a way that certain components of the displacement field are continuous across adjacent element interfaces. In the case of plates and shells the polynomials must be able to ensure continuity of the appropriate rotation terms. The

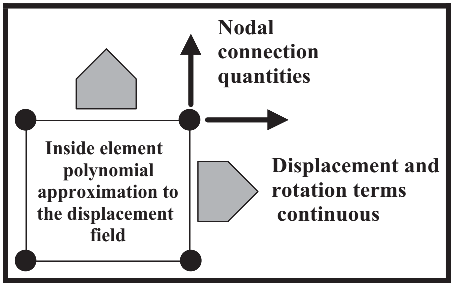

Figure 1.1 Schematic of a displacement element.

polynomials are then defined in terms of nodal values that can be specified at a vertex, as shown in Figure 1.1, or at specific points along element edges or surfaces. Adjacent elements are now connected to each other through these nodes and because of this the nodal displacements or rotations are called connection quantities. Loads are applied to the finite element model through these same nodes. It is worth noting that the displacement finite element formulation degenerates the structure under analysis into a set of points distributed through the space occupied by the structure and there is no longer any explicit representation of the actual structure nor any explicit representation of the physically distributed load system. As shown in Chapters 2 and 3, the resulting nodal model is then solved in terms of the initially unknown nodal connection quantities and terms such as element stresses are derived from this solution.

The formulation for an equilibrium finite element is similar, in principle, to that of the displacement element as shown in Figure 1.2. In the case of an equilibrium element, the interior stress field is approximated by polynomials and the connection from one element to the next is via side or surface forces that are distributed along the element edges and surfaces. As shown in Chapter 4, the displacement

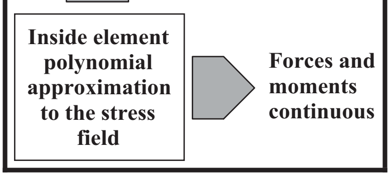

Figure 1.2 Schematic of an equilibrium element.

formulation gives rise to stress discontinuities across the inter-element interface boundaries. Equilibrium elements, on the other hand, generate continuous stresses as the solution crosses from one element to an adjacent element but inter-element compatibility is not preserved. In the early days of finite element analysis this stress continuity property made equilibrium elements popular with aircraft stressmen in allowing them to track the internal load paths. This property was also mistakenly thought to mean that equilibrium elements were more accurate than displacement elements. An early example of the use of equilibrium elements can be found in a publication by one of the pioneers of the Finite Element Method, Fraeijs de Veubeke, and his gifted assistant Guy Sander in reference [7]. A very good description of this type of element can be found in the book by Tong and Rossettos [8] which is, unfortunately, now out of print.

Reference [8] is also a good starting point for a description of the third type of finite element, known as the hybrid element, which also receives a brief description in reference [6]. This element is shown schematically in Figure 1.3 where it can be seen that there are two fields being deployed for the element. Inside the element it looks like an equilibrium element but there is also a line distribution of displacement along the edge of the element shown in Figure 1.3 or a surface distribution if a three-dimensional solid element is employed. This additional displacement field is approximated by either a one- or two-dimensional polynomial depending on the dimensionality of the element. This approximation is formed in terms of nodal displacement or rotation values which then form the connection quantities for attaching adjacent elements. The element

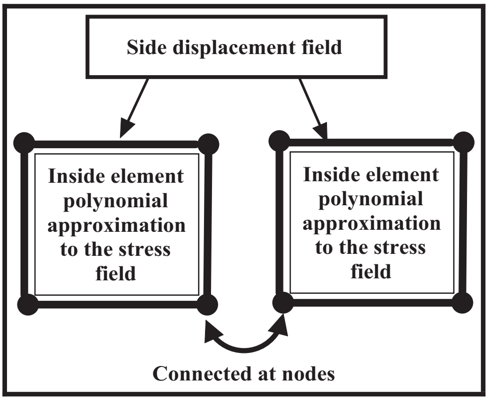

Figure 1.3 Schematic of a hybrid element.

appears to the outside world as a standard displacement element. The displacement fields are playing a subtle role as they act as Lagrange multipliers on the continuity condition that element stress equilibrium is maintained across element boundaries as with the equilibrium element. These elements find application in the development of finite elements for plate and shell analysis problems.

1.3 FINITE ELEMENT ANALYSIS AND FINITE ELEMENT REPRESENTATIONS

A finite element analysis is a numerical simulation of the behaviour of a real-world structure which is intended to provide information that can be used by a designer or design team to ensure a structural design is fit for purpose when it enters in-service operation. The process of setting up an analysis requires that such factors as the loads applied to the structure, the structural behaviour and responses, the boundary conditions, etc., are all represented by a set of mathematical functions or operations. This is an important concept to understand because the focus of the analysis is the real world, which is *not* a mathematical model.

A finite element analysis is, therefore, a process that takes an actual structure, subject to its constraints including attachments to other structures – including the Earth which is simply a very big structure. It then has to perform the following tasks:

1. Convert the real-world system into a mathematical description.
2. Turn this description into a form which allows a computer to be brought into the picture to solve this mathematical problem.
3. Take this output and turn it back into parameters that relate to the real-world structural behaviour.

In undertaking this series of operations the finite element analysis can be envisaged as passing through a series of 'worlds' or, more accurately, representations. Although these representations and their ramifications are covered in detail in later chapters, it is worth setting the scene in this introductory chapter.

The first of these is the *Real-World* representation which constitutes the object (structure) which is to be analysed and its environment. It is a representation because the structure itself often does not physically exist at the time of the analysis. Nevertheless this representation models the in-service structure and the way that it will be actually loaded, supported, etc., and the way that it responds to the loading and support environment. For the purposes of the present book it is assumed that there are no errors associated with this representation, even though this may not be the case in many instances.

The *Reduced Real-World* representation is the first level of abstraction and is a modified version of the real world in which uncertainties concerning the real structure and its environment are taken into account. The level of abstraction may be quite large such that the loads may not be the actual loads seen by the structure, nor the structural form of the real structure. Often this type of abstraction is imposed by the qualification requirements which attempt to account for uncertainties in, for example, the loads acting on the structure by imposing stereotype loads together with safety factors.

The third representation is the *Idealised World* which takes the structural world model and turns it into a form which can be analysed by the Finite Element Method. This is a very profound level of abstraction which converts the structural model with its welds, rivets, bounded joints, etc., into a smooth model in which each component, together with its boundary condition, loading situation, etc., can be mathematically defined. Thus the decisions concerning factors such as the linearity or otherwise of the structural behaviour are made at this stage. This is the most critical part of the whole finite element analysis process as, in a loose sense, the construction of an idealised world represents a transition from a world 'exterior' to the computer to an 'interior' world.

Once the idealisation process has been performed a number of closely related representations are constructed. First is the *Finite Element World* which maps onto the idealisation a set of specific finite elements which can adequately represent the mathematical behaviour defined in the idealised world. This also includes the selection of the element boundary conditions and the element loads. Second is the creation of a *Meshed World* in which the elements have a specific location, shape, etc. Finally in this sequence of 'interior' worlds is the *Solution World* where a procedure is employed to obtain a solution for the idealised structure represented by the idealised world.

As far as the analysis process is concerned, the final stage transforms the results expressed in the solution world and reinterprets them so that they provide results for the structural world problem. This is often called *De-Idealisation.* The results obtained from this process can be used to correlate with the structural tests performed when the structure has, eventually, been constructed. This book is endeavouring to create a methodology that ensures no significant differences appear when this correlation is performed.

1.4 MULTI-MODEL ANALYSES

There are many reasons why the evaluation of a structural design might require the use of several finite element models with different levels of fidelity. If the structure is completely new, it is necessary to get some feel for its basic characteristics if only, for no other reason, to undertake an initial sizing of the structural components and parameters. The starting point could be a low-fidelity analysis model with relatively few finite elements or low-order elements. This provides an initial 'view' of the structure, identifying major load paths, inertia characteristics, the mechanisms through which externally applied loads enter the structure, etc. Once the initial configuration has been established, higher fidelity models can come into play where either more or different elements – or, indeed, both – are employed. This first move up the fidelity ladder could be a transitional step where the analyst 'zooms in' on certain parts of the structure using local high-fidelity models, leaving the rest of the structure to be handled by the initial low-fidelity model. A further step could be the development of one or more high-fidelity models for the entire structure. In the case where the structural design is based on an existing design this same pathway may be required depending on the complexity of the structure as discussed in Chapter 5. The number and type of finite element models required depend on many factors with each analysis focused on clear objectives.

Irrespective of the number of analyses required, the levels of representation outlined in Section 1.3 are present in all of them. This situation provides ample opportunity for the introduction of errors so that the results from the final or intermediate analyses are significantly different from the measured performance of the in-service structure. Thus error control procedures are required for all analyses during the build-up process for the results.

1.5 CONSISTENCY, LOGIC AND ERROR CONTROL

The process of analysing a structure has to start with a clear definition of the real-world design problem and end with a finite element model that accurately reflects the behaviour of structure when in operational use and subject to the in-service loads. In seeking to control or eliminate the differences between the prediction from the finite element simulation and the actual operational performance it is necessary to employ a process-based approach. This process has to be both consistent and reliable if it is to inspire confidence. The case for such an approach can be argued, as done from a consistency viewpoint following the earlier work in reference [9] or from a reliability viewpoint as done in reference [10].

Several implications are associated with the concept of consistency. First, there is the requirement that the analysis process can be decomposed into a logical and coherent set of steps. These must give rise to a sequence of operations that provide a linked pathway whereby information controlling the analysis is passed from one step to its successor and backwards through any required feedback loops. Thus, the logical sequence which the method must follow is defined in a consistent manner. Second, consistency requires error control; if the errors cannot be controlled, the method cannot be consistent. Lack of error control would mean that an identical problem could be solved on two separate occasions and produce different results. Finally, and following on from the second point, a consistent method for performing finite element analyses should demonstrate repeatability with respect to results produced for a specific problem when run at different times by different analysts.

The second aspect associated with consistency identifies the need for the creation of a finite element error control and error treatment methodology. This is difficult, particularly when the complete analysis process is taken into account. Appealing to the Church–Turing theorem establishes that it is perfectly reasonable to ask that a computer program *perfectly* simulate the behaviour of a physically realisable system – such as a structure subject to loads. Thus, if the elastic behavioural response of the actual real-world structure, under static loading conditions, at a finite number of points is represented by a stiffness matrix K_R and a finite element model with n elements by K_n, the theorem implies that a measurable error e_n exists such that:

$$K_R = K_n + e_n$$

and that the error can be made vanishingly small for an appropriate choice of K_n. The corollary to the theorem is that a process must, in principle, exist where the difference between the response of a real-world structure and that of a finite element model of this structure can be controlled. This principle would imply that, while in practical terms it is impossible to reduce the error to zero, it should be possible to provide absolute bounds on e_n.

There are two problems associated with this principle. The first is the question of uncertainties. The implication in writing the real-world stiffness matrix K_R is that a unique real-world structure exists against which the finite element result is being compared. Unfortunately this is not the case since all the attributes of an actual structure are subject to variability: variability in material properties, fabrication, loading, etc. Such variability, discussed in Chapter 5, is termed uncertainty when applied to a new structure at the design stage and cannot be accurately assessed. Thus uncertainty constitutes a residual term in e_n which cannot be computed and, thus, this term cannot be driven to zero. As we shall see later, it is possible to consider ways in which uncertainties can be introduced into a formal error assessment process.

The second of these two problems concerns the difficulty of choosing an error control mechanism which avoids violating Gödel's theorems. The first of these theorems essentially states that if a formal theory, in which proofs are expressible by mathematical formulae, is proved to be consistent, it is not possible to prove completeness. The second asserts the impossibility of proving the consistency of such a theory by methods 'formalisable within the theory'. The essence of these theorems, in the present context, is that error bounds cannot be achieved by using finite element results to self-reference. This poses particular difficulties for methods proposed in this book which are attempting to create error control and error bounding methods for a new structure which, at the time the analysis is performed, does not exist. While the Church–Turing theorem tells us that a finite element model of the (as yet non-existent) structure under design must exist, Gödel's theorems imply that we have no adequate way of deciding how that model should be constructed using unaided finite element data. Thus, adequate error bounding procedures cannot be obtained using finite element information only. The questions raised in this section are clearly linked to questions of computability which are discussed by Belytchko and Mish in reference [11].

The arguments introduced in this section underpin the rationale for creating a consistent method to undertake finite element analyses of real-world structures. The process to achieve this is introduced in later chapters and represents one approach to the problem ensuring analyses are accurate and repeatable. In some ways it can be argued that what is advanced is simply common sense. However, common-sense solutions only become common after they have been explained. Other expositions of approaches for controlling error propagation in a finite element analysis are available. One is the SAFESA$^{\mathrm{TM}}$ method introduced in the Preface, another is in the excellent report, reference [12], from NASA examining technologies for use in the analysis of the Space Shuttle's external tank.

1.6 CHAPTER CONTENTS

Most technical books are not read as a novel where it is essential to start at the beginning and proceed chronologically to the final chapter; rather, the reader selects those parts relevant to the technical issues being addressed. To assist the reader in making a judgement as to where relevant information can be found we can now describe what will be found in the other chapters of this book.

1.6.1 Chapter 2 Overview of Static Finite Element Analysis

1.6.1.1 Aim

To go through the entire process followed by a computer in solving a structural analysis problem using the Finite Element Method for statically loaded problems exhibiting a linear response.

1.6.1.2 Outline

Chapter 2 covers the entire process starting with the derivation of the individual element stiffness matrix, coordinate transformation, assembly of elements into the global stiffness matrix, the application of boundary conditions, solution of the problem at the global level and then the evaluation of element properties (e.g. stress). It uses a set of spring elements as the initial demonstration example which focuses on a 1-D problem, then moves to considering an assemblage of bar elements for a 2-D demonstration.

Finally, the chapter discusses how the size of a finite element static analysis problem can be reduced. This discussion covers the following:

- condensation
- sub-structures
- symmetry and anti-symmetry.

With the availability of modern computing power, the need to exploit condensation and symmetry/anti-symmetry to reduce the amount of computing time and storage space is often thought to be unnecessary – this is a mistake, it is always worth saving computing effort! Furthermore, the use of substructure techniques is essential when an analysis is being undertaken by a number of separate analysis teams, particularly when these are non-collocated.

1.6.2 Chapter 3 Overview of Dynamic Analysis

1.6.2.1 Aim

To go through the process followed by a computer in solving a finite element analysis problem using the Finite Element Method for dynamically loading structures.

1.6.2.2 Outline

Chapter 3 picks up from Chapter 2 by showing that the introduction of dynamic loads requires the construction of a mass matrix that introduces inertia loads into the analysis employing a single spring element to demonstrate the process. The chapter then shows that the free vibration problem for this simple structure reduces to the solution of an eigenvalue problem. The use of simple checks that assist the analyst in establishing that a robust solution has been found are then highlighted.

Forced responses are then discussed, employing both a modal and direct integration. Both of these solution techniques are developed to include the effects of damping.

Finally, as with Chapter 2, the chapter discusses how the size of a finite element dynamic analysis problem can be reduced. This discussion again covers the following:

- condensation
- substructures
- symmetry and anti-symmetry.

If it is claimed that modern computing power negates the need to exploit condensation and symmetry/anti-symmetry to reduce the size of static analysis problems, this argument cannot be deployed in the case of very large-scale dynamic analysis problems. As with the static analysis case, substructuring is required in the case of multiple analysis teams.

1.6.3 Chapter 4 What's Energy Got to Do with It?

1.6.3.1 Aim

This chapter emphasises the fact that the Finite Element Method, for a statically loaded structure, is actually minimising the potential energy (PE) of the structural system and, for the dynamically loaded structure, the kinetic energy (KE). This demonstrates that the Finite Element Method is a convergent process – at least in an integrated sense.

1.6.3.2 Outline

Chapter 4 begins by defining potential energy and using a simple spring as a demonstration vehicle. It then develops the concept that minimising the PE does, indeed, lead to a correct solution for a statically loaded structure operating in the linear elastic domain. Using a combination of springs, it is shown that using this minimising principle leads to the standard finite element matrix formulation introduced in Chapter 2. The chapter demonstrates that the Principle of Minimum PE is the underlying basis used to create the matrices for the construction of a displacement finite element system focused on the solution of linear static analysis problems. It also introduces the concept of the consistent load vector that allows distributed loads to be accommodated by a set of displacement finite elements that have degenerated the structure to a set of discrete points, distributed across the structural domain.

The discussion of PE concludes with a simple illustration of the use of PE for the generation of appropriate matrices and load vectors for a simple set of statically loaded bar elements. This illustrates:

- that the method converges as the number of elements is increased to the correct potential energy as the number of elements is increased;
- that the consistent method loses loads which are applied at displacement boundaries;
- that there are jumps in stress across the junctions of common elements.

The chapter then addresses the finite element analysis of structures exhibiting dynamic responses. The fact that the solution of this type of analysis problem also requires the minimisation of a specific function, in this case a Lagrangian function that can be directly related to kinetic energy, is illustrated.

The chapter concludes by illustrating the kinetic energy convergence and its relationship to the dynamic response of a structure through the free-vibration analysis of the simple bar structures used in the PE demonstration.

1.6.4 Chapter 5 Preliminary Review of Errors and Error Control

1.6.4.1 Aim

To introduce the reader to some of the basic considerations relating the likely causes of error that can be encountered in a finite element analysis.

1.6.4.2 Outline

Chapter 5 opens with a brief discussion of error and uncertainty and their location in the total analysis process. The concepts of novelty, complexity and experience are introduced and linked to the possibility of the introduction of error and uncertainty within the finite element analysis process. The role of testing within an error control process is discussed. The discussion then moves on to consider the overall qualification process that brings in questions relating to acceptable levels of error and the need for an analysis validation plan.

1.6.5 Chapter 6 Discretisation: Elements and Meshes or Some Ways to Avoid Generated Error

1.6.5.1 Aim

This chapter endeavours to provide simple rules and guidance information to assist an analyst in the selection of appropriate elements and mesh layouts.

1.6.5.2 Outline

Chapter 6 opens with a discussion on using simple rules to work out what a particular element can deliver in terms of stress output which can then be linked to the required level of accuracy. It moves on to consider the use of optimal stress points and the associated concept of superconvergent elements. Meshing issues relating to element shape distortions and element grading are discussed together with some popular abuses. The chapter concludes by indicating how an analyst can attempt to measure and improve the internal level of accuracy.

1.6.6 Chapter 7 Idealisation Error Types and Sources

1.6.6.1 Aim

The chapter discusses the types of errors that can occur in the idealisation process for a finite element analysis of a structure subject to loads that give rise to linear responses.

1.6.6.2 Outline

Chapter 7 covers the range of error types and error sources that can occur in the major stages of a finite element analysis. Although the term error is used extensively in this chapter, many of the 'error' types and sources are due to the presence of uncertainties in the problem definition or model data. The chapter considers error sources due to the need to select a specific form for the structural performance and a domain of analysis. Error sources occurring in the definition of the mathematical model upon which the finite element model will be built are then treated. Finally, the need to control uncertainties in the selection of the boundary conditions and the load definition is explored.

The chapter explains the nature of the error sources, their potential influence on the results and points to ways that these can be identified so that they can be controlled and treated using the methods and approach discussed in Chapter 8.

1.6.7 Chapter 8 Error Control

1.6.7.1 Aim

This chapter introduces methods that can be used to treat the errors and error sources identified in Chapter 7 and other parts of this book.

1.6.7.2 Outline

Chapter 8 approaches the process of controlling error through a hierarchical methodology starting with simple control methods and ending with sophisticated numerical methods. It starts by considering simple engineering-based methods involving ‘hand calculations’ and engineering formulae – these are also used to create an initial ‘view’ of the design problem being confronted. The use of different mathematical models or levels of abstraction to control errors associated with an incorrect selection for the structural model is developed. This is followed by a detailed consideration of sensitivity methods used to bound the impact of any errors or uncertainties, generated during the idealisation process, on the finite element predicted behaviour of the in-service structure. This exploits both direct and indirect sensitivity methods that are derived from methods developed by the structural optimisation community. Both direct and indirect methods are demonstrated for static and dynamic analysis problems.

The chapter lays the foundation for creating a methodology that can be used as the basis of a quality control methodology.

1.6.8 Chapter 9 Error-Controlled Analyses

1.6.8.1 Aim

This is the key chapter of the book. Its aim is to provide a coherent and logical process that allows potential error and uncertainty sources latent within a finite element analysis to be identified and their magnitude assessed and bounded. On a first reading of the book, this chapter should be consulted early in the process as it shows the target application for much of the discussion and methods developed in the first eight chapters of the book.

1.6.8.2 Outline

Chapter 9 commences by addressing some basic questions relating to assessing the fitness for purpose of a finite element system. It then discusses the requirement for the construction of a ‘quality report’ which, ultimately, provides the basis for establishing that an analysis has been performed to an adequate standard.

After these initial sections the chapter describes a multi-level and multi-stage analysis procedure, entitled FEMEC, that starts with a review of a structure (real world or as designed) and moves through an interactive set of processes resulting in the production of analysis results that attempt to predict accurately the behaviour of the structure when in service. This incorporates the processes for identifying and defining error sources introduced in Chapter 7 and links these to the error control and treatment methods from Chapter 8. Each stage is formalised into a number of specific tasks with clear inputs and outputs. The methodology recognises that experience is a distinct advantage as complex problems often require 'lateral vision' when executing stages where judgement is needed in assessing the usefulness or otherwise of the results. In order to encourage the use of experience and lateral vision, the process starts with a preliminary error assessment and progresses through levels of deeper error assessment that often involve feedback loops to earlier stages in the assessment processes.

1.6.9 Chapter 10 FEMEC Walkthrough Example

1.6.9.1 Aim

The aim of this chapter is to provide an illustration of the FEMEC procedure detailed in Chapter 9.

1.6.9.2 Outline

Chapter 10 uses a simple static analysis problem involving a pressure-loaded reinforced plate to illustrate the stages in the FEMEC process. It sets out the design requirements together with the required level of accuracy for the analysis. The process is demonstrated by developing an illustrative full Quality Report for this analysis problem. The chapter concludes by showing a limited application of the methodology to a dynamically responding structure.

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2

Overview of Static Finite Element Analysis

2.1 INTRODUCTION

As explained in Chapter 1, although the existence of equilibrium and hybrid finite element formulations is recognised, this book focuses exclusively on displacement finite elements for reasons explained earlier. Chapter 4 takes a more fundamental approach to the creation of the Finite Element Method but, in this chapter, a very simple approach is taken to the creation of individual element and global matrices. The aim is to provide a background to the Finite Element Method and illustrate the processes used in all computer-based finite element systems. The chapter does not provide a comprehensive development of the Finite Element Method as there are many excellent text books that cover the field, such as those referenced in Chapter 1 and referenced here as [1], [2], [3] and [4]. In addition, a very comprehensive description can be found in the finite element handbook reference [5]. It simply provides the necessary background knowledge to allow the rest of the book to be read without the need to have an additional text book by the side of the reader when employing the methods discussed in the later chapters. In essence, the chapter answers the question ‘How do computers perform a static finite element analysis?’ In answering the question it follows what is known as the ‘direct method’ which exploits matrix analysis that directly mimics the way that the computer operates when addressing a finite element analysis.

A Practical Guide to Reliable Finite Element Modelling A. Morris

2.2 THE DIRECT METHOD FOR STATIC ANALYSES

2.2.1 Element Matrices

The ‘direct approach’ derives element and global stiffness matrices for displacement finite elements for analyses involving structures subjected to static loads, employing the long-established matrix method of structural analysis. It was used by the early pioneers of the Finite Element Method and is a very effective illustration of the method as it follows the steps employed by finite element computer codes when solving a problem. For more information, the reader can consult one of the early classics in the field cited at reference [6]. A more detailed description that still exploits matrix theory but provides a great deal more information on the computational processes required to support a modern finite element system is given in reference [7].

We can begin very simply by taking a single spring subject to a set of loads as shown in Figure 2.1. The spring has a stiffness denoted by ‘k’ and is subject to a set of forces f_1 and f_2 that give rise to the end displacements u_1 and u_2 at the two nodes 1 and 2.[1]

Applying Hooke’s Law, the relationship between the force f_1 and the displacements u_1 and u_2 is given by:

$$f_1 = ku_1 - ku_2 \tag{2.1}$$

Similarly:

$$f_2 = ku_2 - ku_1 \tag{2.2}$$

Combining these two simple equations into a matrix formulation gives:

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{2.3}$$

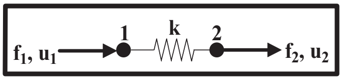

Figure 2.1 Single loaded spring.

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[1] For an isolated spring in static equilibrium, such as the one shown here, the nodal forces f_1 and f_2 would be equal and opposite.