FUNCTIONAL ANALYSIS
An Introduction to Banach Space Theory

Terry J. Morrison

Pure and Applied Mathematics
A Wiley-Interscience Series of Texts, Monographs, and Tracts

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PURE AND APPLIED MATHEMATICS

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An Introduction to Banach Space Theory

TERRY J. MORRISON

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St. Peter, Minnesota

A Wiley-Interscience Publication

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For Kathy and Cheri

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PREFACE

> *A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories; and one can imagine that the ultimate mathematician is one who can see analogies between analogies.*
>
> —*S. Banach*

The theory of Banach spaces really began with the 1922 publication of Stefan Banach's doctoral dissertation "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales" in ***Fundamenta Mathematicae***, followed in 1932 by his famous monograph *Théorie des Opérations Linéaires* in Warsaw, Poland. In the minds of a majority of mathematicians, the appearance of these two publications also signaled the onset of "modern" functional analysis as an independent discipline. Through the influential work of Banach, M. Fréchet, J. Hadamard, H. Hahn, F. Hausdorff, D. Hilbert, S. Mazur, J. von Neumann, F. Riesz, and M.H. Stone, to name but a few, mathematics was changed; there was no looking back.

It is hoped that by studying the ideas and techniques presented in this text, by following through on the directions indicated by many of the fundamental results presented here, and by gaining a deeper understanding of the beauty and subtlety underlying most of Banach's work and legacy, you will develop an appreciation for and understanding of this rich area of mathematics. Much is to be gained from mastering the basic ideas you will be exposed to in the material that follows, and in then continuing to pursue both the theoretical avenues they open and the many applications they represent in mathematics and science. Consider this book a beginning point only.

What I have attempted to do here is to gather and organize the work of those mathematicians that has formed the basis for the discipline of functional analysis as it is known today. There is, of course, some arguably fundamental material omitted from this book; you are naturally seeing my personal bias as to what is most important and most memorable. However, you will be exposed to the basic ideas, techniques, and methods that form the underpinnings of this discipline. Primarily through the study of Banach spaces (with the occasional side trip into general topological vector spaces), you should gain the necessary tools and insight to successfully investigate whatever area of mathematics you have chosen (or which has chosen you). Ideally, you will be sufficiently excited and motivated to make Banach spaces themselves an integral part of your mathematics.

A FEW NOTES TO THE STUDENT

This book is meant to be more than just a reference source, a repository of definitions and theorems; it is meant to be read. In the commentary between theorems I have attempted to help motivate you and explain why certain results are important, when particular attention should be paid to their method of proof, and when further exploration beyond the text itself is recommended or even needed. I have often included more detail than one might expect to see in a book at this level; on the other hand, you will note a fair number of familiar phrases such as "the details are left to the reader." These have been chosen with some purpose in mind; more detail where the author feels initial proding and aid in reaching the heart of the argument is necessary, less when you should be adequately prepared to proceed on your own. My goal is for you to gain understanding and insight from the presentation; I hope the sometimes less formal nature of the arguments will help, and not hinder, this process.

You will soon realize that there are no exercises at the end of sections, as you will find in most books; don't be misled, however, there are exercises and problems embedded in the text itself. I have chosen to present these "in context," as the results they give often have immediate benefit. Other times, as when having to recreate the context in which they are found could be distracting, they are where they belong most naturally. These problems, while they are not delineated or numbered in any special way, should be recognizable when you meet them, and are introduced by such phrases as ". . . as the student should verify . . . ," or ". . . it is straightforward to see . . . ," or ". . . as a moment's thought reveals . . . ," each such phrase a clue to the student. There are also more explicit exercises and problems whose solution has been left to you, and it is expected you will "fill in the missing details." It is the author's intention that all these exercises be completed and the ideas internalized. Understanding does not come passively.

A FEW NOTES TO THE INSTRUCTOR

The book is designed for a two-semester first course in functional analysis and should allow time for topics presented here to be explored in further detail, or new material to be introduced if desired. The introductory material is for the convenience of the student; Chapter 1 is where it all begins. While not all the examples of the first chapter need be presented, there are many ideas surrounding them to which the student should be exposed. While it is not strictly necessary that all topics in every chapter be presented and thoroughly understood by the students before proceeding to the next, the author feels the majority of the material plays an integral role in what every young analyst should know and master.

ACKNOWLEDGMENTS

The author is deeply indebted to many people who helped (and sometimes literally forced) this book to come into existence. Undoubtedly, the two most important and influential are my best friend, largest supporter in every possible meaning of that word, partner, most successful motivator, and wife, Kathy; and, my advisor, mentor, and long-time friend, Dr. Professor Joseph Diestel of Kent State University. There is absolutely no question that you would not be reading this now without their presence in my life; my appreciation and gratitude are beyond words.

There are, of course, many others whose suggestions, ideas and remarks, and general support mathematically as well as personally during the long development of this book have been greatly appreciated, and who must be acknowledged; particularly all my colleagues in the Mathematics Department at Gustavus Adolphus College; I must especially mention R. Rietz, J. Rosoff, and L. Hewitt. Additionally, my thanks and appreciation are extended to M. DiBattista, N. Miller, M. Gaviano, G. Georgacarakos, R. Hilbert, D. Kelley, T. Henry, and R.H. Lohman. Thanks must also be given to my former colleagues at Addis Ababa University, and particularly my students there, who were subject to early versions of the lecture notes that eventually led to this manuscript, especially Fereja Tahir and Negash Begashaw, who have become close friends and colleagues. Acknowledgment must also be given to the staff at John Wiley & Sons, particularly Lisa Van Horn, Heather Haselkorn, Andrew Prince, and Steve Quigley, for their support and help in making this process as smooth and easy as possible.

Finally, a special debt must be acknowledged to B.J. Pettis who, late one night over a mostly empty bottle of Virginia Gentlemen some years ago, advised me that if I was going to dedicate my dissertation to my wife, not to, but to wait and save that for my first real book; while it has taken some time, I have done so.

St. Peter, Minnesota TERRY J. MORRISON
October, 2000

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FUNCTIONAL ANALYSIS

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INTRODUCTION

Before entering into the formal material of the text, there are always ground rules that need to be understood so that the reader and the author know they will progress down the same path and understand they share the same goal. While the direction of the road and recognition of the purpose in this case are not difficult to comprehend (after all, the student assumes the author is devoted to presenting and explaining the fundamental concepts of the discipline to the uninitiated, and the author assumes the reader is approaching the material eager to learn, motivated to pursue the ideas beyond the meager boundaries of the text itself, and is more than adequately prepared for this journey), all involved realize this works well theoretically. More often than not misunderstandings arise early: How in the world can they expect we've seen *this stuff* before?, "as the reader should recall . . ." and "from the student's real analysis course, one has . . ." become roadblocks (or at least at times substantial barriers) to achieving the desired ends.

While there is no practical way to completely avoid these problems, since any two different readers have not only different backgrounds but different purposes for reading this text, they can to some extent, be alleviated. At least this is the author's purported rationale behind including these opening comments, remarks, and observations before formally engaging in the adventure that follows; and it is hoped it will be an adventure, a remarkable experience. After all, contained herein are some of the basic ideas and techniques that lead to the theory of Banach spaces, one of the most beautiful and profound disciplines in all of mathematics, certainly within the realm of functional analysis. What more need be said?

Beyond the reasonable expectation that students wishing to partake of this material have been exposed to the standard material one normally encounters in any appropriate sequences in real analysis and topology, the author tries to assume only a fledging mathematical maturity and the openmindedness to give beauty and subtleness a chance to work its magic when encountered. You don't have to be a true believer to begin with; this will be a natural outgrowth of the exposure to the work of Stefan Banach and his followers.

Of course, practically speaking, assumptions must be made and reliance on background must be assumed. While what immediately follows is not by any means all-inclusive, the author has chosen a few short topics that may aid readers as they begin their perusal of these topics. They need not be considered in the order given, or for that matter closely considered at all. They are included because the author assumes there are always a few facts and ideas whose understanding will help to ease the transition into the new material, and that the reader may or may not have within easy reach.

The first short section on notational conventions and standard symbology used throughout the text should, of course, be quickly scanned. The first topic presented is a brief look at product spaces at a very basic level. This can more than likely be ignored until one is ready to begin Chapter 3 and perhaps even then; its inclusion here is to ensure that the reasons behind changing topologies on a given space are understood and realized to be reasonable in the given context. The second short section concerning finite-dimensional spaces is ideally totally unneeded; the notion of a Hamel basis and the role it plays in understanding these spaces are indistinguishable algebraically and topologically, underlies much of what we will encounter in the near future. The only possible exception is the inclusion here of a pretty result by F. Riesz on approximations in a normed linear space, which often proves useful to have at hand.

The last material in the Introduction will prove to be superfluous to any student with a firm grounding in abstract measure and integration theory. Because the Daniell approach to integration theory is not necessarily a standard topic in real analysis, it is included here for those readers not fortunate enough to have been exposed to it. While undoubtedly insufficient to completely prepare the student for its use in Chapter 1, it is hoped the brief exposure to the basic concepts will allow the reader to achieve an adequate facility with the central ideas to either be able to fill in the missing details, or at least locate and prepare for its use.

So, we begin.

NOTATION AND CONVENTIONS

As always whenever one encounters a new book or set of notes, there are notations and conventions that the author uses, but with which the student is not familiar, and, of course, these can lead to confusion. We include a number of the more common of these here; other more specialized notation will be encountered in the text as they are needed. A comprehensive list of symbols and notation can also be found in the appropriate index at the end of the text.

We start with a list of some basics.

1. We will use the following designations for some standard collections of numbers:

$\mathbb{R}$ for the real numbers;
$\mathbb{C}$ for the complex numbers;
$\mathbb{S}$ for an arbitrary scalar field (when the specific use of either $\mathbb{R}$ or $\mathbb{C}$ is immaterial);
$\mathbb{N}$ for the natural numbers $\{1, 2, 3, \ldots\}$;
$\mathbb{I}$ for the collection of all integers.

Furthermore, we will always use the symbol θ to denote the zero vector in an arbitrary space rather than the number 0 so that no confusion arises.

2. Anytime we are considering a singly indexed object such as a sequence, a sum, or a limit (as long as the underlying index set is countable; that is, essentially the natural numbers), if the beginning index value does not matter or is unimportant to the meaning of the expression, this value will be omitted in our representation. Similarly, as typically the upper limit of our indexing is ∞, we will omit this as well and write these expressions as follows:

$$(x_n)_n \quad \text{or} \quad \sum\nolimits_n a_n \quad \text{or} \quad \lim_n y_n$$
$$\text{instead of} \quad (x_n)_{n=1}^{\infty} \quad \text{or} \quad \sum\nolimits_{n=1}^{\infty} a_n \quad \text{or} \quad \lim_{n\to\infty} y_n$$

In case we wish to allow our index set to be uncountable (or at least not restrict ourselves to only countable sets), we will change the symbol used for our underlying index set and assume it is some arbitrary directed set Γ. Thus, the expressions just listed will usually be written as

$$(a_\gamma)_\Gamma \quad \text{or} \quad \lim\nolimits_\Gamma y_\gamma \quad \left[\text{or perhaps} \quad (a_\gamma)_{\gamma\in\Gamma} \quad \text{or} \quad \lim\nolimits_{\gamma\in\Gamma} y_\gamma\right]$$

to distinguish this. In other words, we consider $(a_\gamma)_\Gamma$ to be a net.

In a similar manner, to indicate that a function (or operator) f is the pointwise limit of the sequence $(f_n)_n$, we write $f = \lim_n f_n$, rather than $f(x) = \lim_n f_n(x)$ for all x, when no confusion should arise.

3. Anytime we wish to make it overtly clear the choice of a particular constant depends upon a previously determined value, we subscript that constant with the dependent value; that is,

"... there exists an $N_\varepsilon \in \mathbb{N}$ such that ..."

means that the choice of the integer N depends on the value of ε given, while

"... choose $\delta_{n,\varepsilon} > 0$ with ..."

means that the value of δ depends on the values of both n and ε, and so forth.

4. The usual symbols for set operations will be used, so if we let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a (nonempty) family of sets indexed by the directed set Γ, then:

$$\bigcup_{\gamma\in\Gamma} A_\gamma = \{a : a \in A_\gamma \text{ for some } \gamma \in \Gamma\},$$
$$\bigcap_{\gamma\in\Gamma} A_\gamma = \{a : a \in A_\gamma \text{ for some } \gamma \in \Gamma\},$$
$$\prod_{\gamma\in\Gamma} A_\gamma = \{(a_\gamma)_\Gamma : a_\gamma \in A_\gamma \text{ for each } \gamma \in \Gamma\}.$$

We will further use $A\backslash B$ to denote the set-theoretic difference of A and B, that is, $A\backslash B = \{x : x \in A \text{ but } x \notin B\}$, and A^c to denote the complement of the set A (relative to some universal set, of course).

5. Another convention that will be followed is the occasional use of the symbol $\equiv$ in place of the usual $=$. This will be employed primarily to stress that we are defining a particular object by this description. That is, if we wish to *define* the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(3x)$, we might write "let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) \equiv \sin(3x)$" to further emphasize that f is being defined here. This will often be used when we define an object in the middle of a proof or in an explanatory paragraph where it is inconvenient to interrupt the flow of the text. It should also aid readers in realizing they are encountering a particular object for the first time, and have not inadvertently overlooked its meaning.

6. In conjunction with remark 5, the reader should note that definitions or descriptive titles that will be used and referred to through the text are often embedded in an explanation, remark, or proof of a statement. In order to make these easier to recognize and locate for later reference, they will be written in **boldface lettering** to improve visibility. As with any special symbols used, all can be found in the symbol or subject index.

7. As the final comment, the student should note that in order to more clearly specify the actual end of a proof, we will always use the symbol ■. No special meaning should be attached to this symbol, as it merely serves as the obvious visual indicator of the end of an argument.

PRODUCTS AND THE PRODUCT TOPOLOGY

While products and the product topology are not difficult concepts, but are standard fare in any course in general topology, somehow it often appears to be one of those topics quickly lost at the conclusion of the course itself. While it is certainly assumed the readers are capable of reexamining some of this material on their own when it is necessary, we include here a short overview of this material as both a ready reference and particularly to remind the student of the concept of weak topologies. A good grasp of these general ideas will serve the reader well when we encounter them in Chapter 3.

Of course, at this point there is no reason, *a priori*, to simply read through the material that follows. Scan it now, remember it is here, and then return for a closer look should this become necessary at some point in the future. In any case, we now give a quick walk through the basic ideas.

Let Γ be an arbitrary directed set and for each $\gamma \in \Gamma$, let $(\mathcal{X}_\gamma, \tau_\gamma)$ be a topological space. The **Cartesian product** (or just **product**) of the family $\{(\mathcal{X}_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ is the set of all functions $f: \Gamma \to \bigcup_{\gamma\in\Gamma} \mathcal{X}_\gamma$, where for each $\gamma \in \Gamma$ we have $f(\gamma) \in \mathcal{X}_\gamma$. For notational purposes, we will denote this collection of functions by $\Pi_{\gamma\in\Gamma}\mathcal{X}_\gamma$ or by $\Pi_\Gamma \mathcal{X}_\gamma$ when this will cause no confusion. It is easy to see that there is another way to view the elements of a product; namely, each element $x \in \Pi_\Gamma \mathcal{X}_\gamma$ can be realized as a net in the following way:

$$x \in \prod\nolimits_\Gamma \mathcal{X}_\gamma \quad \text{if and only if} \quad x = (x_\gamma)_{\gamma\in\Gamma} \text{ where } x_\gamma \in \mathcal{X}_\gamma \text{ for every } \gamma \in \Gamma.$$

Hence *statements about the elements of an (arbitrary) product space are automatically statements about nets.*

In addition to the preceding remarks, we employ the following conventions both to simplify the notation used and because the reader is likely to encounter (or has encountered) these same concepts in other books and contexts where they are frequently found:

1. If the family Γ is countable, say we have $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n, \ldots$, then we will write the product as $\Pi_n \mathcal{X}_n$ to conform with our other notations.
2. If for each $\gamma \in \Gamma$ we have $\mathcal{X}_\gamma = \mathcal{X}$ (that is, all the spaces $\mathcal{X}_\gamma$ are the same space $\mathcal{X}$), then we will write the product $\Pi_\Gamma \mathcal{X}_\gamma$ as $\mathcal{X}^\Gamma$; often as $\mathcal{X}^\omega$ should $\Gamma = \mathbb{N}$.
3. Finally, in the case that we have a finite product of spaces that are all the same, say $\mathcal{X}_i = \mathcal{X}$ for $i = 1,2, \ldots, n$, we will write $\Pi_{i=1}^n \mathcal{X}_i = \mathcal{X}^n$.

We now indicate how to topologize the product $\Pi_\Gamma \mathcal{X}_\gamma$ of a number of topological spaces.

For each $\gamma_0 \in \Gamma$, define the map $\pi_{\gamma_0} : \Pi_\Gamma \mathcal{X}_\gamma \to \mathcal{X}_{\gamma_0}$ by $\pi_{\gamma_0}(f) = f_{\gamma_0}$. That is, π_{γ_0} is the map that selects the γ_0th-coordinate of f, for each f in the product. Each such map π_{γ_0} is called the **γ_0-coordinate map** or the **γ_0-projection map**.

By definition (that is, we are defining it here) the **product topology** on $\Pi_\Gamma \mathcal{X}_\gamma$ will be the weakest topology on the product for which each of the maps π_γ, for $\gamma \in \Gamma$, is continuous, that is, it is the smallest topology for which all of the projection maps are continuous.

It can be seen that a **local base** of open sets for the product topology is the family of all sets of the form $\bigcap_{k=1}^n \pi_{\gamma_k}^{-1}(U_{\gamma_k})$, where each set U_{γ_k} is a τ_{γ_k}-open set in $\mathcal{X}_{\gamma_k}$, for $k = 1,2, \ldots, n$ and $n \in \mathbb{N}$. That is, each such set is a finite intersection of inverse images of open sets under the (appropriate) projection map. Thus, if $f \in \Pi_\Gamma \mathcal{X}_\gamma$, then a **fundamental system of neighborhoods** of f is the family of all sets of the form:

$$\{g \in \prod_\Gamma \mathcal{X}_\gamma : g(\gamma_k) \in \mathrm{U}_{\gamma_k} \text{ for } k = 1,2,\ldots,n, \text{ where } \mathrm{U}_{\gamma_k} \text{ is } \tau_{\gamma_k}\text{-open}\}$$
$$= \bigcap_{k=1}^n \pi_{\gamma_k}^{-1}(\mathrm{U}_{\gamma_k}).$$

At this point it is well worth pausing to consider that there is another way both to "think" about the product topology and to describe it. This is as follows: Take as a base for the open sets of the product all sets of the form $\Pi_\Gamma \mathrm{U}_\gamma$, where

1. for each $\gamma \in \Gamma$, U_γ is open in $\mathcal{X}_\gamma$;
2. for all but finitely many $\gamma \in \Gamma$, we have $\mathrm{U}_\gamma = \mathcal{X}_\gamma$;

that is, sets like $\Pi_\Gamma \mathrm{U}_\gamma$, where $\mathrm{U}_\gamma = \mathcal{X}_\gamma$ for all γ except some finite number $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$. Note that this set can now be written as

$$\prod_\Gamma \mathrm{U}_\gamma = \pi_{\gamma_1}^{-1}(\mathrm{U}_{\gamma_1}) \cap \pi_{\gamma_2}^{-1}(\mathrm{U}_{\gamma_2}) \cap \cdots \cap \pi_{\gamma_n}^{-1}(\mathrm{U}_{\gamma_n})$$
$$= \bigcap_{k=1}^n \pi_{\gamma_k}^{-1}(\mathrm{U}_{\gamma_k}).$$

In other words, we have

$$\begin{aligned} f \in \prod_\Gamma \mathrm{U}_\gamma \quad & \text{if and only if } f(\gamma) \in \mathrm{U}_\gamma \text{ for all } \gamma \in \Gamma; \\ & \text{if and only if } f \in \pi_\gamma^{-1}(\mathrm{U}_\gamma) \text{ for all } \gamma \in \Gamma; \\ & \text{if and only if } f \in \bigcap_\Gamma \pi_\gamma^{-1}(\mathrm{U}_\gamma); \\ & \text{if and only if } f \in \bigcap_{k=1}^n \pi_{\gamma_k}^{-1}(\mathrm{U}_{\gamma_k}) \end{aligned}$$

(since any time $\mathrm{U}_\gamma = \mathcal{X}_\gamma$, then $\pi_\gamma^{-1}(\mathrm{U}_\gamma) = \pi_\gamma^{-1}(\mathcal{X}_\gamma) = \Pi_\Gamma \mathcal{X}_\gamma$).

The following characterization should, by itself, justify our interest in this topology on the product.

Proposition 1. Let $\mathcal{X}$ be a topological space and $f: \mathcal{X} \to \Pi_\Gamma \mathcal{X}_\gamma$, where the product has the product topology. Then f is continuous if and only if $\pi_\gamma \circ f$ is continuous for each $\gamma \in \Gamma$.

Proof. Clearly if f is continuous, so is $\pi_\gamma \circ f$ for every $\gamma \in \Gamma$, since the composition of continuous functions is continuous. On the other hand, suppose that $\pi_\gamma \circ f$ is continuous for all $\gamma \in \Gamma$. Since sets of the form $\pi_\gamma^{-1}(\mathrm{U}_\gamma)$ for $\gamma \in \Gamma$ and U_γ open in $\mathcal{X}_\gamma$ form a subbase for the topology on $\Pi_\Gamma \mathcal{X}_\gamma$, to show that f is continuous, it suffices to show that $f^{-1}(\pi_\gamma^{-1}(\mathrm{U}_\gamma))$ is open for every subbasic open set $\pi_\gamma^{-1}(\mathrm{U}_\gamma)$. But note that this is clear since $f^{-1}(\pi_\gamma^{-1}(\mathrm{U}_\gamma)) = (\pi_\gamma \circ f)^{-1}(\mathrm{U}_\gamma)$ and $\pi_\gamma \circ f$ is continuous by assumption.■

Finally, we now make the connection between the product topology and "weak topologies" a little more explicit and, hopefully, a little clearer. First recall that it is the topology of a space X that determines which functions are continuous on that space (remember, a continuous function is characterized by having inverse images of open sets being open; that is, *being in the topology of X*). Thus the topology one might impose on a particular set determines the continuous functions on that space, or put another way, if you know which functions you *want* to be continuous in advance, then you can (theoretically at least) give X a proper topology so as to ensure your collection has that property. A reasonable question to ask at this point is: Given a space X, what functions might we want to be continuous?

Well, as you might guess, while this depends on the circumstances involved, we can at least give some idea some of the time. For example, suppose X is Euclidean n-space $\mathbb{R}^n$. The most natural set of functions we might want to be continuous is the collection of coordinate functions f_n; that is, the set of functionals that assign to any n-dimensional vector its nth-coordinate. Now, as we all know, the natural topology on $\mathbb{R}^n$ does make each of these continuous; in fact, any function whose range is $\mathbb{R}^n$ is continuous exactly when its composition with each of these coordinate functionals is continuous. Of course, considering what we have said so far about product topologies (and about what you already know about $\mathbb{R}^n$), it should be clear this natural topology on $\mathbb{R}^n$ is exactly the product topology. In fact, it can be shown, without too much work, that this topology is the weakest (or smallest) topology that can be put on $\mathbb{R}^n$, which makes each coordinate functional continuous.

This idea can very naturally be carried a bit further, not only to products of sets besides $\mathbb{R}$, but to arbitrary products of any sets. That is, given a collection of spaces X_γ, for γ in some arbitrary index set Γ, we want to put a topology on the product of the X_γ that will make each of the coordinate (or projection) maps π_γ continuous, this being as natural in this context as it is in $\mathbb{R}^n$. Accordingly, the product topology on these sets is called the weakest topology for which each projection map is continuous, a natural enough idea.

So, what we basically have is that in the setting of taking a product of spaces, where the maps we want to be continuous are fairly evident, the product topology *is* the weakest topology we can put on the product making each of the given maps continuous. But suppose now we are in a setting where we have a space X (not necessarily a product), and a collection of functions on X (perhaps each mapping into a different space); we should still be able to talk about the weakest topology on X that makes each of these maps continuous, and in fact we can; the definition embedded in the next paragraph makes this explicit.

Suppose we are given a set X and a topological space X_γ with a map $f_\gamma : X \to X_\gamma$ for each $\gamma \in \Gamma$. We will call the **weak topology induced on X by the collection $\{f_\gamma : \gamma \in \Gamma\}$** the smallest topology one can place on X for which each f_γ is continuous. It should be evident that this topology is that for which the sets

$$f_\gamma^{-1}(\mathrm{U}_\gamma) \text{ for } \gamma \in \Gamma \text{ and } \mathrm{U}_\gamma \text{ open in } X_\gamma$$

form a subbase. Clearly, *the product topology on* $\Pi_\Gamma X_\gamma$ *is the weak topology induced by the collection* $\{\pi_\gamma : \gamma \in \Gamma\}$ *of projections*. It should also be clear that Proposition 1 carries over to any weak topology in this sense, without any essential change in the proof. That is, we have the following proposition.

Proposition 2. If $\mathcal{X}$ has the weak topology induced by a collection $\{f_\gamma : \gamma \in \Gamma\}$ of functions $f_\gamma : \mathcal{X} \to \mathcal{X}_\gamma$, then $f : \mathcal{Y} \to \mathcal{X}$ is continuous if and only if $f_\gamma \circ f : \mathcal{Y} \to \mathcal{X}_\gamma$ is continuous for each $\gamma \in \Gamma$.

One final comment here before turning to some of the basics of finite-dimensional spaces. These relatively simple ideas concerning weak topologies will be just as relevant for us in considering normed linear spaces. Here, one of our concerns will be to take a given space $\mathcal{X}$ endowed with the topology it naturally inherits from its norm and seek to find the weakest topology that can be given to $\mathcal{X}$ that still yields its topological dual space remains unchanged. In other words, we want the weakest topology for which all the (norm) continuous linear real-valued functions on $\mathcal{X}$ are still continuous. The full resolution to this search is the primary content of Chapter 3.

FINITE-DIMENSIONAL SPACES AND RIESZ'S LEMMA

Our purpose here is simply to remind the reader of a few of the basic ideas concerning spaces of finite dimension with which they should be familiar; no attempt is made at completeness in any sense. In fact, our primary concern will be to make clear to the reader that linear spaces of the same (finite) dimension are always algebraically isomorphic and that if endowed with a norm, are isomorphic topologically as well. The easiest way to realize this, and one we will look closely at in Chapter 5, comes from considering the concept of a basis in this setting.

Definition 1. Let $\mathcal{X}$ be a (nontrivial) linear space. A collection H of vectors from $\mathcal{X}$ is called a **Hamel basis** (or often just a **basis**) for $\mathcal{X}$ if H is a linearly independent set in $\mathcal{X}$ and the subspace of $\mathcal{X}$ generated by H is all of $\mathcal{X}$ (that is, span $(\mathrm{H}) = \mathcal{X}$).

A particular consequence of the definition itself is that *every element of the space has a unique representation as a (finite) linear combination of basis elements*. That every linear space (regardless of its "dimensionality") has such a basis is a direct result of Zorn's Lemma, whose proof will not be given here. The reader who has somehow missed (or simply misplaced) these results is urged to spend a short time looking at the ideas inherent here; there are some very nice, and eminently useful, techniques that are well worth adding to one's repertoire. While any reasonable linear algebra text will yield such a presentation, one good source is Friedberg et al. (1989).

As indicated earlier, the following well-known result is of some importance to us because of its use and implications for more general normed linear spaces and hence for Banach spaces.

Proposition 3. If $\mathcal{X}$ and $\mathcal{Y}$ are both n-dimensional linear spaces with the same scalar field, then they are (algebraically) isomorphic.

Proof. Letting $\mathbb{S}$ denote the underlying scalar field, we will prove that $\mathcal{X}$ (and hence $\mathcal{Y}$ as well) is isomorphic to $\mathbb{S}^n$ (and so isomorphic to each other by the symmetric and transitive nature of the "isomorphic" relation).

To see this, let $x_1, x_2, \ldots, x_n$ be a basis for $\mathcal{X}$ so we have that every $x \in \mathcal{X}$ has a unique representation as $x = s_1x_1 + s_2x_2 + \cdots + s_nx_n$, where each s_i is in $\mathbb{S}$. We now define the operator $\mathrm{T}: \mathcal{X} \to \mathbb{S}^n$ by $\mathrm{T}(x) \equiv (s_1, s_2, \ldots s_n)$. It is straightforward to verify that T is linear, bijective, and hence that T^{-1} exists (and is by necessity also a linear bijection). The details we leave to the student.■

Before we give our next result, the student should recall that we often consider linear spaces with a topological structure as well as just an algebraic one. In particular, as previously indicated, this topological structure results from endowing the space with a norm; that is, if $\mathcal{X}$ is a linear space, a **norm** on $\mathcal{X}$ is a mapping $\|\cdot\|: \mathcal{X} \to [0, \infty)$, satisying the properties (i) $\|x\| = 0$ if and only if $x = \theta$ (the zero vector in $\mathcal{X}$); (ii) for any scalar λ and any $x \in \mathcal{X}$, $\|\lambda x\| = |\lambda|\,\|x\|$; and (iii) for any $x,y \in \mathcal{X}$, we have $\|x + y\| \le \|x\| + \|y\|$. In this case, Proposition 3 can be extended to give finite-dimensional spaces that are topologically as well as algebraically isomorphic.

Proposition 4. If $\mathcal{X}$ and $\mathcal{Y}$ are n-dimensional normed linear spaces with the same scalar field, then they are topologically isomorphic.

Proof. Again we show that if $\mathbb{S}$ is the underlying scalar field, then both $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic to $\mathbb{S}^n$, allowing us to conclude our result as before.

So, let $\mathcal{X}$ be an n-dimensional normed linear space over $\mathbb{S}$ with $\{x_1, x_2, \ldots, x_n\}$ a basis for $\mathcal{X}$. Define the operator $\mathrm{T}: \mathbb{S}^n \to \mathcal{X}$ by

$$\mathrm{T}(s) = \mathrm{T}\left((s_i)_{i=1}^n\right) \equiv \sum_{i=1}^n s_ix_i.$$

Then, by Proposition 3, we know that T is a linear (algebraic) isomorphism. That T is continuous follows immediately from the fact that the addition and scalar multiplication operations on such spaces are themselves continuous maps; as the student should verify (for pedagogical reasons, we prove this shortly as Proposition 1.1 in Chapter 1, the needful student may merely "look ahead"), T is continuous. Thus, we need only show that T^{-1} is continuous.

To do this, first note that $S \equiv \{s \in \mathbb{S}^n : \|s\| = 1\}$, being closed and bounded, is compact in $\mathbb{S}^n$ by the Heine-Borel Theorem. Hence T(S) is also compact, and hence closed, in $\mathcal{X}$. Since T is an isomorphism, $\theta \notin T(S)$, and so there must be some open set U, containing θ, such that $U \cap T(S) = \varnothing$. Now choose $\delta > 0$ so that if $V \equiv \{x \in \mathcal{X} : \|x\| < \delta\}$, then $V \subseteq U$. We claim that $V \subseteq T(B)$, where $B = \{s \in \mathbb{S}^n : \|s\| < 1\}$. In fact, if $x \notin T(B)$, then $x = T(z)$ for some $z \in \mathbb{S}^n$, with $\|z\| \geq 1$. Note if $x \in V$, then $x/\|z\| \in V$, which is impossible since

$$\frac{x}{\|z\|} = T\left(\frac{z}{\|z\|}\right) \in T(S)$$

and we know $V \cap T(S) = \varnothing$. Thus it must be that $V \subseteq T(B)$. But then we have $T^{-1}(V) \subseteq B$ and $T^{-1}((1/\delta)V) \subseteq (1/\delta)B$. From this, in turn, we get $(1/\delta)V = \{x \in \mathcal{X} : \|x\| < 1\}$, and so T^{-1} maps the open unit ball in $\mathcal{X}$ into a bounded set in $\mathbb{S}^n$ (thus every bounded set of $\mathcal{X}$ into a bounded subset of $\mathbb{S}^n$) and hence T^{-1} is continuous, as was needed.■

For the sake of completeness here, you should note (and naturally be able to verify) the following useful facts all come directly from this theorem:

1. In finite-dimensional normed linear spaces, closed bounded sets are always compact.
2. Any finite-dimensional normed linear space is complete.
3. In a general normed linear space, all finite-dimensional linear subspaces are closed.

As a final result in this section we present the very general and useful classic result of F. Riesz, now known as Riesz's Lemma; sufficiently interested readers should see Riesz (1918) for his original presentation. You will note that it is not restricted to finite-dimensional spaces at all, but holds in any normed linear space setting. We will initially encounter its use when we first consider the idea of approximation in Chapter 4.

Theorem 1 (Riesz's Lemma). Let $\mathcal{X}$ be a normed linear space and $\mathcal{X}_o$ be a proper closed subspace of $\mathcal{X}$. Then for each real number α with $0 < \alpha < 1$, there is an $x_\alpha \in \mathcal{X}$ such that $\|x_\alpha\| = 1$ and $\|x - x_\alpha\| \geq \alpha$ for all $x \in \mathcal{X}_o$.

Proof. Let x_1 be any element of $\mathcal{X} \backslash \mathcal{X}_o$ and let $d \equiv \inf\{\|x - x_1\| : x \in \mathcal{X}_o\}$. Since $\mathcal{X}_o$ is closed, we know that $d > 0$. Now, since $(1/\alpha)\, d > d$, we know there is some $x_o \in \mathcal{X}_o$ such that $\|x_o - x_1\| \leq (1/\alpha)\, d$. For notational purposes, we will let $h \equiv \|x_o - x_1\|^{-1}$, and choose $x_\alpha = h\,(x_o - x_1)$. Then $\|x_\alpha\| = 1$, and if we let $x \in \mathcal{X}_o$, so is $h^{-1}x + x_o$, and so

$$\|x - x_\alpha\| = \|x - hx_1 + hx_o\| = h\,\|(h^{-1}x + x_o) - x_1\| \geq h\, d.$$

But, $h\,d = \|x_o - x_1\|^{-1} d \geq \alpha$ by the way in which x_o was chosen, and so $\|x - x_\alpha\| \geq \alpha$ for all $x \in X_o$, and we are done.■

We can restate Riesz's Lemma as: *Given any closed, proper subspace X_o of a normed linear space X, there exist on the surface of the unit ball of X points whose distance from X_o is as near to* 1 *as we wish.* This will prove useful to recall later in these notes.

One should note that while we can always find points as *close* to a distance of 1 from X_o as we want, it is not true that we can necessarily find points on the surface of the ball whose distance from X_o is *exactly* 1. A relatively simple example of this can be found in the space $C([0,1])$ of continuous real-valued functions on [0,1] [see, for example, Taylor (1958)].

THE DANIELL INTEGRAL

Typically, when we think of developing a general theory of integration, we fall back on our experience in dealing with the Lebesgue integral, or perhaps even the Riemann integral. We begin with some notion of the measure of nice sets, and then extend this idea to include a more complex collection of sets on which we have a structure that allows us some control over their interrelationships. We then balance this with wanting to have a sufficiently rich collection of sets so that our notion of measure is not only "natural" in some sense, but allows the flexibility we will need to accomplish our goals; that is, be able to work with the broad range of functions we will need or want to be able to "integrate". Of course, this naturally leads us to considering what collection of functions our notion of measure will need or be able to "handle", and of what natural idea of integration will be compatible with these restrictions.

Thus, often enough to make the point at least a valid one, we begin with measuring sets in such a way it directly generalizes our elementary notion of length, use this notion to generate a broad class of functions that will be "nice" (i.e., measurable) in this context, and then develop an integration theory consistent with our old Riemann and Lebesgue integrals and hope for the best.

Instead, what we will do here—and the rationale for including this material in this text at all—is a little different and may not have been encountered by the typical student. We will begin with some kind of simple or "elementary" integral defined on a small collection of "elementary functions" and then work toward enlarging this set of functions (and consequently extending our integral) to larger collections in such a way that the result has all the properties we want to be able to retain from the Lebesgue integral. Of course, coming with this will be a resultant concept of measure, so that eventually we arrive at the same end as before.

The first person who really successfully carried out this process was the English mathematician P.J. Daniell, who did most of his work in the early 1900s, and thus the basic integral obtained in this fashion is usually called the Daniell integral (or Daniell functional). The development we sketch next, while far from all-encompassing, will roughly follow his lead. The reader should note that there is an assumption here that full details need not be given and that general ideas together with indications of techniques will be sufficient either to allow the reader to supply the missing justifications on their own, or find the motivation to seek more comprehensive and complete developments. Such can be found, for example, in the well-written *Real Analysis* by Royden (1963), whose general presentation we follow here.

We begin by introducing the appropriate setting and putting forth some fundamental definitions. You should note that our immediate focus is on a fairly general collection of functions over which we have a moderate degree of control and on how we might define their integral in a reasonable fashion; ideally, the rest will follow.

Let Ω be a set and L be a family of real-valued functions defined on Ω closed under finite linear combinations and with $f \vee g$ ($\equiv \max\{f,g\}$) and $f \wedge g$ ($\equiv \min\{f,g\}$) in L whenever f and g are in L (you should recall that such a family is called a **vector lattice**). While this may initially seem to be a very abstract collection of functions to be concerned with, it is easy to see that any linear space L of functions is a vector lattice, provided we require $f \vee 0$ to be in L whenever f is (just note $f \vee g = (f-g) \vee 0 + g$, while $f \wedge g = f + g - (f \vee g)$). So, those linear spaces of functions that include the "positive" part of each of their members, or $f^+ \equiv f \vee 0$ for each f, are always vector lattices. Of course, as $|f| = f^+ + (-f)^+$, these spaces are closed under the taking of absolute values. On the other hand, if we have a vector space L with $|f| \in$ L when $f \in$ L, it is always a vector lattice, as f^+ can be realized as $\frac{1}{2}(f + |f|)$, so these restrictions are the same.

Now let I be a (real-valued) linear functional on L. Here I is called **positive** if $I(f) \geq 0$ for each nonnegative function $f \in$ L. Note that such functionals always preserve the order of L; that is, if $f \leq g$, then $I(f) \leq I(g)$. A positive linear functional I on L is called a **Daniell integral** (or sometimes a **Daniell functional**) if it satisfies the following condition:

> If $(f_n)_n \subseteq$ L is increasing with $f(\omega) < \lim_n f_n(\omega)$ for all $\omega \in \Omega$, then $I(f) \leq \lim_n I(f_n)$.

Since we will use them interchangably, it is worthwhile to note, and the student should certainly verify, that this condition is equivalent to the following property:

> If $(f_n)_n \subseteq$ L is a sequence of nonnegative functions with $f(\omega) < \sum_n f_n(\omega)$ for all $\omega \in \Omega$, then $I(f) \leq \sum_n I(f_n)$.

There are, of course, many examples of Daniell integrals we are already familiar with. For example, we can let L be the family of all continuous functions on $\mathbb{R}$ that vanish outside of some finite interval, and take I to be the

Riemann integral. Or, we can let L be the family of all simple functions defined with respect to a given measure μ, with I being the natural integral (i.e., $I(s) = \Sigma_{i=1}^{n} \alpha_i \mu(A_i)$, where $s = \Sigma_{i=1}^{n} \alpha_i \chi_{A_i}$).

More generally, in the preceding example we could take L to be all integrable functions with respect to the measure μ provided we take care that the sum of two functions is not defined at points where we would get $\infty - \infty$. (To guarantee this, just require $f+g$ to be in L whenever it is well-defined, and $I : \mathrm{L} \to \mathbb{R}$ to be a mapping satisfying $I(\alpha f) = \alpha I(f)$ and $I(h) = I(f) + I(g)$ whenever $h = f + g$). A positive linear functional I on L, in this sense, is called a **Daniell integral** if it satisfies any one of the (equivalent) conditions given earlier.

What we would like to do is extend our basic Daniell integral to a wider class of functions than just L itself; namely, to the collection of all extended real-valued functions on Ω that can be realized as the limit of a monotone increasing sequence of functions from L. If we denote this class of functions by L_e, it should be clear that L_e is a vector lattice. As for any increasing sequence $(\varphi_n)_n$ from L, $(I(\varphi_n))_n$ is an increasing sequence in $\mathbb{R}$ and so has limit (or is $+\infty$), we define the integral any f in L_e to be $I(f) = \lim_n I(\varphi_n)$. It is straightforward to verify that $I(f)$ depends only on f itself, not on the sequence $(\varphi_n)_n$.

Thus, we have extended the Daniell integral to the family L_e on which it preserves the order of L_e and satisfies $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for $\alpha, \beta \geq 0$ and $f, g \in \mathrm{L}_e$. Moreover, if f is any nonnegative function, with $(\varphi_n)_n$ an increasing sequence from L with f as their limit, by replacing each φ_n by $\varphi_n \vee 0$, we can assume each φ_n is nonnegative. Also by letting $\psi_1 \equiv \varphi_1$ and $\psi_n \equiv \varphi_n - \varphi_{n-1}$ for $n > 1$, we have $f = \Sigma_n \psi_n$ and that $I(f) = \lim_n I(\varphi_n) = \lim_n I(\Sigma_{k=1}^{n} \psi_k) = \lim_n \Sigma_{k=1}^{n} I(\psi_k) = \Sigma_n I(\psi_n)$, so that $f \in \mathrm{L}_e$ exactly when there are nonnegative functions $(\varphi_n)_n \subseteq \mathrm{L}$ with $f = \Sigma_n \varphi_n$. Of course, with this we get not only that the sum of any sequence of nonnegative functions from L_e is still in L_e, but that $I(\Sigma_n f_n) = \Sigma_n I(f_n)$.

The extension of the Daniell integral to an arbitrary function on Ω is now pretty standard: one defines the upper and lower Daniell integrals by $\bar{I}(f) \equiv \inf\{I(g) : g \geq f \text{ and } g \in \mathrm{L}_e\}$ and $\underline{I}(f) \equiv -\bar{I}(-f)$), and declares a function on Ω to be **Daniell integrable** whenever $I(f) = \bar{I}(f) = \underline{I}(f) < \infty$; the class of all Daniell integrable functions on Ω is denoted by L_1. Of course, we need to know we have a legitimate extension of I to the collection L_1, and the basic properties listed below guarantee this. As their verification is straightforward, we leave these to be supplied by the student:

1. $\bar{I}(\alpha f + \beta g) \leq \alpha \bar{I}(f) + \beta \bar{I}(g)$ for $\alpha, \beta \geq 0$;
2. Both $\bar{I}$ and $\underline{I}$ preserve order;
3. $\underline{I}(f) \leq \bar{I}(f)$, and they agree on $f \in \mathrm{L}_e$;
4. $\bar{I}(\Sigma_n f_n) \leq \Sigma_n \bar{I}(f_n)$ for any nonnegative functions $(f_n)_n$ on Ω; (given $\varepsilon > 0$, for each n just choose $g_n \in \mathrm{L}_e$, with $g_n \geq f_n$ and $I(g_n) \leq \bar{I}(f_n) + \varepsilon/2^n$).

It should be noted that L_1 is a vector lattice; while this is not difficult to see (one just shows that L_1 is a linear space with $f^+ \in \mathrm{L}_1$ for each $f \in \mathrm{L}_1$), it involves enough manipulation to be a bit messy. We will avoid the technicalities here.

Not surprisingly, the Daniell integral does satisfy appropriate versions of the major convergence theorems enjoyed by integrals arising from more standard measures. Again, we will not give full details here, but try to indicate the primary direction and idea needed. We begin with the Daniell counterpart to the monotone convergence theorem.

Proposition 5. If $(f_n)_n$ is an increasing sequence in L_1 with $f = \lim_n f_n$, then $f \in L_1$ exactly when $\lim_n I(f_n) < \infty$; in this case $I(f) = \lim_n I(f_n)$.

Proof. The necessity of the condition is easily seen, sufficiency follows from letting $g \equiv f - f_1$, and noting $g = \Sigma_n(f_{n+1} - f_n)$ so that we have $\bar{I}(g) \leq \lim_n I(f_n) - I(f_1)$. This immediately yields $I(f) \leq \lim_n I(f_n)$, with the other half of the inequality following from noting $\underline{I}(f) \geq \lim_n I(f_n)$.■

Fatou's lemma in the Daniell context is given by the following proposition.

Proposition 6. If $(f_n)_n \subseteq L_1$ are nonnegative, then $\inf_n f_n \in L_1$ with $\underline{\lim}_n f_n \in L_1$ whenever $\underline{\lim}_n I(f_n) < \infty$. In this case $I(\underline{\lim}_n f_n) \leq \underline{\lim}_n I(f_n)$.

Proof. If, for each $n \in \mathbb{N}$, we let $g_n \equiv f_1 \wedge f_2 \wedge \cdots \wedge f_n$, then $(g_n)_n \subseteq L_1$, which decreases to $\inf_n f_n$. Noting $(-g_n)_n$ increases to $-\inf_n f_n$, a moment's reflection on Proposition 5 yields $\inf_n f_n \in L_1$. Now, for each n, setting $h_n \equiv \inf\{f_k : k \geq n\}$ gives us a (nonnegative) sequence in L_1 increasing to $\underline{\lim}_n f_n$, which must be in L_1, as $\lim_n I(h_n) \leq \underline{\lim}_n I(f_n) < \infty$. The final inequality is immediate from Proposition 5.■

The final result in this context is just the Lebesgue dominated convergence theorem, which also tells us I really is a Daniell integral on L_1.

Proposition 7. Let $(f_n)_n \subseteq L_1$ with, for all $n \in \mathbb{N}$, $|f_n| \leq g$ for some $g \in L_1$. Then $I(\lim_n f_n) = \lim_n I(f_n)$.

Proof. As in the standard case, this proposition follows almost directly from Proposition 6. Just note that $(f_n + g)_n$ is a nonnegative sequence in L_1 with $I(f_n + g) \leq 2I(g)$, so that $\lim_n f_n + g \in L_1$, with $I(\lim_n f_n + g) \leq \underline{\lim}_n I(f_n) + I(g)$. Of course, we now have $I(\lim_n f_n) \leq \underline{\lim}_n I(f_n)$. Since $(g - f_n)_n$ is also a nonnegative sequence with $I(g - \lim_n f_n) \leq I(g) - \overline{\lim}_n I(f_n)$, $I(\lim_n f_n) \geq \overline{\lim}_n I(f_n)$ gives us all we need.■

The last result that we need here is due to Marshall Stone. In some ways it pulls all of this together and ties our new Daniell integral back to the integration theory developed in the more standard manner. In order to see this, we must first introduce the concept of measurable functions, and of measures, as they arise in this context.

Thus, we say a nonnegative function f on Ω is **measurable** (with respect to I) if $f \wedge g \in \mathrm{L}_1$ for all $g \in \mathrm{L}_1$. Because L_1 is a vector lattice, it follows that $f \vee g$ and $f \wedge g$ are measurable whenever f and g are nonnegative measurable functions; from this it is not hard to see that $\lim_n f_n$ is measurable as long as the f_n are. Measurable (and integrable) functions naturally give rise to sets with these properties; in fact, $\mathrm{E} \subseteq \Omega$ is called a **measurable set** if its characteristic function χ_{E} is measurable, and an **integrable set** if χ_{E} is integrable. It is not difficult to see (and a straightforward exercise to prove) that provided Ω is measurable (that is, 1 is a measurable function), the collection of all such measurable sets is a σ-algebra.

We make one final comment before taking the last step we really need here. Suppose our measurable sets are a σ-algebra and f is a nonnegative integrable function. Note if $\alpha \le 0$, then $\mathrm{E} = \{\omega : f(\omega) > \alpha\} = \Omega$ and so is a measurable set. If $\alpha > 0$, consider the function $g \equiv (1/\alpha)f - ((1/\alpha)f \wedge 1)$, and note we have $g \in \mathrm{L}_1$ with $g(\omega) \ge 0$ for all $\omega \in \mathrm{E}$. Since it should be clear that $(1 \wedge ng)_n$ is a sequence from L_1 increasing to χ_{E}, we have that χ_{E} is measurable so that $\mathrm{E} = \{\omega : f(\omega) > \alpha\}$ is measurable (for any $\alpha \in \mathbb{R}$) as well.

So, measurable functions? Measurable sets? There should be a measure somewhere in sight. Well, let's define the set function μ on the σ-algebra of measurable sets by

$$\mu(\mathrm{E}) = \begin{cases} I(\chi_{\mathrm{E}}) & \text{if E is integrable} \\ \sup\{\mu(\mathrm{A}) : \text{A is integrable with } \mathrm{A} \subseteq \mathrm{E}\} & \text{otherwise.} \end{cases}$$

It should be clear that $\mu(\varnothing) = 0$ and that $\mu(\mathrm{A}) \le \mu(\mathrm{B})$ for A and B integrable sets, so it must hold for measurable sets as well. If $\mathrm{E} = \bigcup_n \mathrm{E}_n$, where the E_n are pairwise disjoint measurable sets, then given any integrable $\mathrm{A} \subseteq \mathrm{E}$, if we let $\mathrm{A}_n \equiv \mathrm{A} \cap \mathrm{E}_n$, then each A_n is integrable and, by Proposition 5, $\mu(\mathrm{A}) = \Sigma_n \mu(\mathrm{A}_n) \le \Sigma_n \mu(\mathrm{E}_n)$ so that $\mu(\mathrm{E}) \le \Sigma_n \mu(\mathrm{E}_n)$. On the other hand, if $\mu(\mathrm{E}) < \infty$, then given $\varepsilon > 0$, for each n we can find an integrable $\mathrm{A}_n \subseteq \mathrm{E}_n$ with $\mu(\mathrm{A}_n) > \mu(\mathrm{E}_n) - \varepsilon/2^n$. But this means that $\mu(\mathrm{E}) \ge \Sigma_n \mu(\mathrm{A}_n) > \Sigma_n \mu(\mathrm{E}_n) - \varepsilon$, so that $\mu(\mathrm{E}) \ge \Sigma_n \mu(\mathrm{E}_n)$. As this inequality holds even if $\mu(\mathrm{E}) = \infty$, we actually have that μ is countably additive and hence is a measure as we wanted.

We are now ready for the main result toward which we have been laboring; namely, the beautiful theorem of M.H. Stone that tells us that the natural integral with respect to this measure μ is exactly the Daniell integral I on L_1.

Theorem 2 (Stone's Theorem). Let L be a vector lattice of functions on a set Ω with the property that $f \in \mathrm{L}$ implies that $1 \wedge f \in \mathrm{L}$, and let I be a Daniell integral on L. Then there is a σ-algebra Σ of subsets of Ω, and a measure μ on Σ, such that each f on Ω is integrable with respect to I if and only if it is integrable with respect to μ; moreover, we have $I(f) = \int f \, d\mu$.

Proof. Note that by the preceding discussion, we have that the family Σ of measurable sets with respect to I do form a σ-algebra, and that each nonnegative I-integrable function is measurable on Σ. Since each such I-integrable function is the difference of two nonnegative I-integrable functions, every I-integrable function must be measurable on Σ. Moreover, if we define μ as in the last paragraph, and let f be any nonnegative I-integrable function, then for any n and k in $\mathbb{N}$ we have $E_{k,n} \equiv \{\omega \in \Omega : f(\omega) > k/n\}$ is measurable. Also, since

$$\chi_{E_{k,n}} = \chi_{E_{k,n}} \wedge (n/k) f$$

$\chi_{E_{k,n}} \in L_1$ with $\mu(E_{k,n}) < \infty$. If we now let $g_n = (1/n)\Sigma_{k=1}^{n^2} \chi_{E_{k,n}}$ for each $n \in \mathbb{N}$, then $(g_n)_n \subseteq L_1$ is an increasing sequence that converges pointwise to f, so that $I(f) = \lim_n I(g_n)$. But

$$I(g_n) = \frac{1}{n}\sum_{k=1}^{n^2} I(\chi_{E_{k,n}}) = \frac{1}{n}\sum_{k=1}^{n^2} \mu(E_{k,n}) = \int g_n \, d\mu.$$

Since $\int f \, d\mu = \lim_n \int g_n \, d\mu$, Proposition 5 yields $I(f) = \int f \, d\mu$ and f is integrable with respect to μ. Finally, as any f that is I-integrable is the difference of two nonnegative I-integrable functions, f is also μ-integrable with $I(f) = \int f \, d\mu$ as desired. The difficult half of our result now holds.

On the other hand, let f be any nonnegative function on Ω integrable with respect to μ. As before, we construct the sets $E_{k,n}$ and the functions g_n, and note that since each $E_{k,n}$ has finite measure, each $g_n \in L_1$. But then Proposition 5 yields that we have $f \in L_1$ [after all, $(g_n)_n$ increases to f and $\lim_n I(\varphi_n) = \int f \, d\mu < \infty$!], that is, f is integrable with respect to I.■

As a final comment, it is worth noting that it is not all that difficult to show that if we take Σ to be the smallest σ-algebra for which each $f \in L_1$ is measurable, then the measure μ corresponding to each Daniell integral is unique.

1
BASIC DEFINITIONS AND EXAMPLES

In this chapter we present many of the fundamental examples of Banach spaces that should serve as an indication of the type and broad range of spaces we will be concerned with throughout most of the remainder of this book. While we will, in fact, encounter some other examples and some arguably rather basic results in later chapters, most of what are now considered to be the elementary "classic" Banach spaces are contained herein.

The chapter is divided into two sections: the first presents the basic examples of the spaces themselves, as well as some of the standard fundamental ideas and results we will need in our subsequent work; the second section is devoted primarily to calculation of the "dual space" (or space of continuous linear scalar-valued functions defined on the original space) of many of our examples from the first section. The importance of understanding the relationship between a space and its dual will initially become most apparent beginning in Chapter 3, where we first begin to really develop some of the deeper consequences of this relationship. For now, however, we begin building our collection of concrete spaces and elementary facts.

1.1 EXAMPLES OF BANACH SPACES

While it should be clear that the reader of this text should already have a passing familiarity with normed linear spaces, without belaboring the point we begin with the relevant definitions and ideas.

Definition 1.1. Let $\mathcal{X}$ be a linear space (that is, a vector space). By a **norm** on $\mathcal{X}$ we will mean a mapping $\|\cdot\| : \mathcal{X} \to [0,\infty)$ such that

(i) $\|x\| = 0$ if and only if $x = \theta$ (the zero vector in $\mathcal{X}$);

(ii) $\|\cdot\|$ is **positive homogeneous**; that is, for any scalar λ and any $x \in \mathcal{X}$, we have $\|\lambda x\| = |\lambda|\,\|x\|$;

(iii) $\|\cdot\|$ is **subadditive**; that is, for any $x,y \in \mathcal{X}$, we have $\|x + y\| \le \|x\| + \|y\|$.

In this case, the pair $(\mathcal{X},\|\cdot\|)$ is called a **normed linear space**. Sometimes, if it is clear what the norm is for a particular space $\mathcal{X}$ (such as the norms defined for the spaces in all of Examples 1.1–1.12 in this section), or if $\mathcal{X}$ is just an arbitrary normed linear space with whatever generic norm "$\|\cdot\|$" $\mathcal{X}$ must have, we will drop any specific representation of the norm and just refer to the pair $(\mathcal{X},\|\cdot\|)$ as the normed linear space $\mathcal{X}$.

Before considering our first examples, it is important to notice that all normed linear spaces $(\mathcal{X},\|\cdot\|)$ have the more familiar property of being metric spaces. That is, one can always induce a metric structure on $\mathcal{X}$ via the formula $\rho(x,y) \equiv \|x - y\|$. Thus, this functional ρ as just given, is a well-defined metric on the linear space $\mathcal{X}$ [so that $(\mathcal{X},\rho)$ is always a metric space, as the student should be able to readily verify for himself]. This leads us to the next definition.

Definition 1.2. We say that a normed linear space $(\mathcal{X},\|\cdot\|)$ is a **Banach space** whenever the metric space $(\mathcal{X},\rho)$ derived from $(\mathcal{X},\|\cdot\|)$ as before is a complete metric space.

Thus, *in a Banach space $\mathcal{X}$, unless otherwise noted, convergence is always with respect to the metric induced by the particular norm for $\mathcal{X}$.*

Before proceeding further, we give some examples and simple consequences of these definitions. We start with a list of basic, and what should be familiar, examples.

Example 1.1: $\mathbb{R}$ and $\mathbb{C}$. The real number system, $\mathbb{R}$, with its usual linear structure and norm defined to be "absolute value" is a Banach space, as the student should know from any typical undergraduate analysis text. Likewise, the complex numbers, $\mathbb{C}$, together with its usual linear structure and absolute value for a norm, constitutes a complex Banach space. (We shall agree to use the word "complex" whenever the underlying scalar field of a Banach space is specifically the complex number system, otherwise, all Banach spaces referred to are real Banach spaces.)

Example 1.2: $\mathbb{R}^n$ and $\mathbb{C}^n$. Let $n \in \mathbb{N}$, then **Euclidean *n*-space**, $\mathbb{R}^n$, with the usual linear operations of vector addition and scalar multiplication together with the **Euclidean norm** for $x = (x_1,x_2,x_3,\ldots,x_n) \in \mathbb{R}^n$ defined by

$$\|x\| = \|(x_1,x_2,x_3,\ldots,x_n)\| \equiv \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

is a Banach space that is sometimes denoted by the symbol l_2^n (note that here, the 2 refers to the "l_2-norm", which will be defined in Example 1.9, while the n refers to the dimension of the space and signifies that we are looking at some sort of finite-dimensional space). In proving that $\mathbb{R}^n$ with this norm is a Banach space, the only seemingly difficult step is in establishing Property (iii) of the norm. While the student should either already know or be able to verify this directly, we will postpone our proof until Example 1.9, where it will be established in a more general setting. The completeness of the Euclidean spaces comes, of course, from the fact we know convergence here is just "coordinatewise" convergence and, since $\mathbb{R}$ itself is complete, verifying that this property holds should represent no problem. One should note that similarly, $\mathbb{C}^n$, **complex Euclidean *n*-space**, is also a Banach space.

Our next example, standard fare in any reasonable beginning analysis course, is given here for good reason. The basic idea and techniques presented serve as a model (in fact, often with only minimal modification) for how to deal with a variety of similar, yet important, examples that will be encountered later. While this particular example is very specific, as it is worked through, the student should look for places where the specificity is not really used. Thinking about what properties of the underlying real interval and of the actual collection of functions defined on that interval actually possess and of how these are really used to achieve the desired results will provide an insight into not only the following example, but into a number of other spaces, beginning with Example 1.7. Grasping this general idea of deducing more abstract-oriented information from particular more familiar examples is one that will not only arise often in one's mathematical career, but often provides the basis on which valuable research is conducted.

Example 1.3: $C([0,1])$. The space $C([0,1])$ of continuous real-valued functions (or, complex-valued functions) on the interval $[0,1]$ with the usual linear operations of pointwise addition and scalar multiplication is a Banach space when endowed with the norm: $\|f\|_\infty \equiv \sup\{|f(t)| : 0 \le t \le 1\}$ for $f \in C([0,1])$. (Recall that we know this supremum exists since any continuous function defined on a compact set, such as $[0,1]$, is bounded, and hence $\|f\|_\infty < \infty$ for any such f.)

To see that this is indeed a norm as claimed, first note that Properties (i) and (ii) of the norm are readily verified for $\|\cdot\|_\infty$. To establish Property (iii), let f_1 and f_2 be in $C([0,1])$ and observe that for each $t \in [0,1]$,

$$|(f_1 + f_2)(t)| = |f_1(t) + f_2(t)| \le |f_1(t)| + |f_2(t)|.$$

Now note that by definition of the norm, we have $|f_1(t)| \le \|f_1\|_\infty$ and $|f_2(t)| \le \|f_2\|_\infty$, so that for each t, $|(f_1 + f_2)(t)| \le \|f_1\|_\infty + \|f_2\|_\infty$. From this it follows that

$$\|f_1 + f_2\|_\infty = \sup\{|(f_1 + f_2)(t)| : 0 \le t \le 1\} \le \|f_1\|_\infty + \|f_2\|_\infty$$

and Property (iii) is established.

Finally, to see that $C([0,1])$ is a Banach space, we must show that Cauchy sequences in $C([0,1])$ (with respect to the metric induced by $\|\cdot\|_\infty$) converge in $C([0,1])$. So, let $f_n \in C([0,1])$ for $n \in \mathbb{N}$, and suppose that $\|f_n - f_m\|_\infty \to 0$ as $n,m \to \infty$. We will first construct what turns out to be the "appropriate" function $f: [0,1] \to \mathbb{R}$, then show that our sequence $(f_n)_n$ actually converges to this function, and finally conclude with showing this function is in $C([0,1])$.

To this end, we first observe that since $(f_n)_n$ is a Cauchy sequence, there is for each $\varepsilon > 0$ a natural number N_ε such that $|f_n(t) - f_m(t)| \le \varepsilon/3$ for all $n,m \ge N_\varepsilon$ and all $t \in [0,1]$. In particular, for each fixed $t \in [0,1]$ $(f_n(t))_n$ is a Cauchy sequence of *real numbers* and hence convergent. Now define for each $t \in [0,1] : f(t) \equiv \lim_n f_n(t)$ and observe that for all $t \in [0,1]$, $|f(t) - f_m(t)| = \lim_n |f_n(t) - f_m(t)| \le \varepsilon/3$ whenever $n,m \ge N_\varepsilon$. From this it follows that given $\varepsilon > 0$, there is a positive integer N_ε such that for $m \ge N_\varepsilon$

$$\|f - f_m\|_\infty = \sup\{|f(t) - f_m(t)| : 0 \le t \le 1\} \le \varepsilon/3 < \varepsilon$$

so that $(f_m)_m$ converges (uniformly, in fact!) to f. We need only show the continuity of f to be through. So, using the same N_ε as before, for $n = N_\varepsilon$, we have by the continuity of each f_n that given $t \in [0,1]$ there exists a neighborhood U_t of t such that if $s \in U_t$, then $|f_n(t) - f_n(s)| \le \varepsilon/3$. From this we have that if $s \in U_t$, then

$$\begin{aligned} |f(t) - f(s)| &\le |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)| \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

and thus f is continuous at t (for each t in $[0,1]$). The proof of this example is compete.

One should note, before going to the next example, that embedded in this proof is the fact that *the uniform limit of a sequence of continuous functions is itself continuous.*

Example 1.4: $C(\mathcal{K})$. If one looks carefully at the previous example, it will be noticed that the set $[0,1]$ itself was explicitly used very little except in defining the norm; essentially only the inherent properties of the set were employed. Consequently, if we substitute for $[0,1]$ any Hausdorff topological space $\mathcal{K}$ that is compact, we have that $C(\mathcal{K})$, with its usual linear pointwise operations and norm defined by

$$\|f\|_\infty \equiv \sup\{|f(k)| : k \in \mathcal{K}\}$$

is a Banach space. Of course, if we consider the collection of all continuous, complex-valued functions on $\mathcal{K}$, then $(C(\mathcal{K}), \|\cdot\|_\infty)$ constitutes a complex Banach space.

Before proceeding with any further examples, we will study a few of the basic properties of normed linear spaces. This first fact shows us that in a normed linear space, the topological structure (determined by the norm on the space), and the linear structure (given by the vector space nature of X itself) blend together nicely. It also gives rise naturally to a more general class of spaces, which include Banach spaces, and which, for us, it will be necessary to study and understand in greater detail. As a thorough comprehension of these spaces is not really needed until we begin looking at duality theory in Chapter 3, here we describe only briefly what they are and what specifically distinguishes them from Banach spaces.

Theorem 1.1. Addition and scalar multiplication are always continuous maps for any normed linear space; that is, the maps $\varphi : X \times X \to X$ and $\psi : \mathbb{S} \times X \to X$ defined by $\varphi(x_1, x_2) \equiv x_1 + x_2$ and $\psi(\lambda, x) \equiv \lambda x$ are continuous maps for any normed linear space X.

Proof. A quick thought about the definition of continuity together with a careful choice of constants yields our theorem as an easy consequence of the following two inequalities:

$$\|\varphi(x_1, x_2) - \varphi(\bar{x}_1, \bar{x}_2)\| = \|(x_1 - \bar{x}_1) + (x_2 - \bar{x}_2)\| \le \|x_1 - \bar{x}_1\| + \|x_2 - \bar{x}_2\|$$

and

$$\begin{aligned} \|\psi(\lambda_1, x_1) - \psi(\lambda_2, x_2)\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\ &\le \|\lambda_1 x_1 - \lambda_1 x_2\| + \|\lambda_1 x_2 - \lambda_2 x_2\| \\ &= |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| \|x_2\|. \quad \blacksquare \end{aligned}$$

Guided in part by this result, we can make the following definition, which distinguishes these kinds of "nice" topologies from others.

Definition 1.3. Let X be a linear space endowed with a topology τ. We call τ a **linear topology** on X whenever the operations of addition and scalar multiplication are continuous functions from $X \times X$ and $\mathbb{S} \times X$ into X, respectively. In order to adhere to historical distinctions, if τ is also a T_1-topology (i.e., if singleton sets are closed in X), then the pair (X, τ) (or sometimes it is written $X(\tau)$) is called a **linear topological space** (or a **topological vector space**).

Thus, another way to state Theorem 1.1 is that *every normed linear space is a topological vector space* (although the converse to this is false, as a little thought about topologies and what it might take to be "normable" should quickly reveal).

Using Theorem 1.1, an easy induction argument (which we leave to the student) will now yield a proof of our next fact.

Proposition 1.1. In a topological vector space $(\mathcal{X},\tau)$, all (finite) linear combinations of the scalars $\lambda_1,\lambda_2,\ldots,\lambda_n$ and vectors $x_1,x_2,\ldots x_n$ determine continuous linear mappings of the product space $\Pi_{i=1}^n\mathbb{S}\times\Pi_{i=1}^n\mathcal{X}$ into $\mathcal{X}$.

Moreover, the mappings ψ and φ from $\mathcal{X}$ to $\mathcal{X}$ itself and defined by $\psi(x)\equiv\lambda x$ (for a fixed scalar $\lambda\neq 0$) and $\varphi(x)\equiv x+y$ (for a fixed vector y in $\mathcal{X}$) are homeomorphisms of $\mathcal{X}$ onto itself. (Recall that this means both ψ and φ are one-to-one, onto, continuous mappings with a continuous inverse.)

A particular consequence of the continuity of these linear operations is our next fact. The student should pay close attention to the proof itself, as we will encounter this same technique in the future.

Theorem 1.2. The closure of a linear subspace $\mathcal{Y}$ of a topological vector space $(\mathcal{X},\tau)$ is still a (closed) linear subspace of $\mathcal{X}$.

Proof. Given a subset A of $\mathcal{X}$, let us use the notation cl(A) to denote the closure of the set A in the topology τ.

Now, consider the map $\xi:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ given by $\xi(x_1,x_2)\equiv\lambda_1x_1+\lambda_2x_2$ for fixed scalars λ_1,λ_2. Note that since $\mathcal{Y}$ is a linear subspace of $\mathcal{X}$, we have

$$\mathcal{Y}\times\mathcal{Y}\subseteq\xi^{-1}(\mathcal{Y})$$

and further that $\xi^{-1}(\mathcal{Y})\subseteq\xi^{-1}(\mathrm{cl}(\mathcal{Y}))$. Moreover, since ξ is continuous, $\xi^{-1}(\mathrm{cl}(\mathcal{Y}))$ is a closed set and thus $\mathrm{cl}(\mathcal{Y})\times\mathrm{cl}(\mathcal{Y})\subseteq\xi^{-1}(\mathrm{cl}(\mathcal{Y}))$. From this it readily follows that $\mathrm{cl}(\mathcal{Y})$ is also "closed" under the operations of addition and scalar multiplication, and hence is a linear subspace of $\mathcal{X}$ as claimed.■

We now continue with our examples by presenting a common way in which new normed linear spaces arise from known spaces.

Example 1.5. Let $\mathcal{X}$ be any normed linear space and $\mathcal{Y}$ be a linear subspace of $\mathcal{X}$. Then note that the restriction of the norm on $\mathcal{X}$ to $\mathcal{Y}$ is clearly a norm on $\mathcal{Y}$. Under normal circumstances, this "restricted" norm is denoted with the same symbols as the original (the usage will thus be clear from context). So, we have that linear subspaces of normed linear spaces are themselves, in a natural way, normed linear spaces.

Again, if we let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ be a linear subspace of $\mathcal{X}$, then by Theorem 1.2 the closure of $\mathcal{Y}$ with respect to the *norm topology* (that is, the topology induced by the metric derived from the norm) is a linear