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Real Analysis

A Constructive Approach

Mark Bridger

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REAL ANALYSIS

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REAL ANALYSIS

A Constructive Approach

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WILEY-INTERSCIENCE
A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data is available.

ISBN-13: 978-0-471-79230-7
ISBN-10: 0-471-79230-6

Printed in the United States of America.

10 9 8 7 6 5 4 3 2

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PREFACE

What is a constructive approach, and why should one take it?

If you look at the table of contents for this book, you'll see mostly familiar topics, but with a slightly different emphasis. There's a long chapter on the real numbers, followed by one on "An Inverse Function Theorem." The chapter on limits, sequences and series is followed by one on *uniform* continuity—why not just *pointwise* continuity? A chapter on the Riemann integral is followed by one on differentiation—but it's actually *uniform* differentiation. All of these departures from the structure of the usual real analysis text result from a careful reassessment of the role of the course in the technical education of undergraduates.

Not every student in Real Analysis is a math major, and, in many schools, only a small percentage of math majors intend to do graduate work in mathematics. A modern course is populated by a wide range of students. Some are headed for careers in secondary education, while there is often a large contingent from the physical sciences and an even larger group from computer science. These students are in the course because they need or want more than a cookbook calculus course. Some need to know more about computability and calculability of floating-point numbers, hence more about the actual nature of the reals. They also need to know about continuity because they need to know about approximations; some need to know about convergence and improper integrals because they need to know about computing special functions and transforms.

But real analysis is not primarily focused on computing. It is, significantly, a course that shapes the way students think about mathematics. Very often it is a student's introduction to precise reasoning and writing.

So. I begin with a careful construction of the real numbers, the field on which most of analysis is played. The approach here, due to Gabriel Stolzenberg, is via intervals of rational numbers and the arithmetic of such intervals. The many elementary theorems about the properties of this arithmetic later reappear as properties of the real numbers, and verifying them provides a gentle introduction to the art and practice of devising and writing readable and correct proofs. Furthermore, there is a useful metaphor: a rational interval is exactly what is obtained when a scientist uses instruments of limited (but known) accuracy to measure something. Families of rational intervals then correspond to multiple measurements, and the condition on a family that any two of its intervals must meet establishes the consistency of its measurements. Finally, a real number is defined to be a family of rational intervals that is consistent in this sense, and that contains intervals of arbitrarily small length. Interval arithmetic, carried over to families of intervals, now becomes real arithmetic, and conditions on the lengths of intervals become the properties of approximation of reals by rationals.

At this point, the students see that the reals have a far more complex structure than the rationals. One important example is the traditional Law of Trichotomy, namely that precisely one of $x < y$, $y < x$, or $x = y$ must hold. This property holds for the rationals, since rational arithmetic is basically integer arithmetic. However, reals can, in general, only be approximated by rationals. Modern computer algebra systems allow the user to specify a tolerance, which is expressed as the number of decimal places. This number can be chosen as large as one pleases, but not infinitely large. The corresponding tolerance, say ϵ, tells us how closely we can distinguish reals using the computer's rational representation. These considerations lead to the formulation of real number comparison that we prove and use throughout the book.

ϵ-Trichotomy *Given any tolerance $\epsilon > 0$, then for any reals x and y, $x < y$, $y < x$, or x and y are within ϵ of each other.*

Thus, a construction of the reals based on rational measurement and an analysis of what we can actually calculate produces a concordance of theory and practice that students of the sciences easily relate to.

Using the notion of ϵ-trichotomy as a tool for comparing real numbers enables us to describe a bisection-like algorithm for finding the inverse of a function f, providing it satisfies upper and lower bounds on its difference quotient $\frac{f(y)-f(x)}{y-x}$. This leads directly to the construction of nth root, exponential and logarithm functions.

Another hallmark of the constructivist program is its emphasis on uniform vs. pointwise continuity:

- *f is pointwise continuous at a if, given any $\epsilon > 0$ we can find a $\delta_a(\epsilon) > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta_a(\epsilon)$.*
- *f is uniformly continuous on S if, given any $\epsilon > 0$ we can find a $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta(\epsilon)$ and $x, y \in S$.*

Uniform continuity on S implies pointwise continuity at each point of S, but the converse is not true: there is no general procedure for *constructing* a single δ from the infinitely many δ_a. Not only is uniform continuity a stronger notion, it is the more desirable version of continuity since it is the one most useful in studying convergence and integrability. It turns out that the usual proofs that the basic functions of analysis are pointwise continuous also prove that these functions are uniformly continuous on appropriate intervals. We exploit this fact from the very beginning, and only use the stronger and more important uniform version of continuity.

We take a similar approach to differentiability. Instead of talking about the derivative of a function at a point, we talk about the derivative *function* on an interval. As with uniform continuity, this notion of uniform differentiability is the one that is of most importance in later theory and applications. In fact, it is an approach that generalizes readily to vector-valued functions of several variables.

An important consequence of using uniform notions is that they produce transparent proofs of important theorems such as the existence of the Riemann integral and the Fundamental Theorem of Calculus.

The pointwise versions of continuity and differentiability do lead to a number of classical examples of functions which are or aren't continuous or differentiable on various dense or nowhere dense subsets of intervals. Since we are emphasizing uniform notions, these examples are relegated to discussions in an appendix and a few exercises, which can be covered at the discretion of the instructor.

In summary, then, this is neither a text in numerical analysis nor one intended solely to prepare students to be professional mathematicians. It is a thoroughly rigorous modern account of the theoretical underpinnings of calculus; and, being constructive in nature, every proof of every result is direct and ultimately computionally verifiable (at least in principle). In particular, existence is never established by showing that the assumption of non-existence leads to a contradiction. By looking through the index or table of contents, you'll see that nothing of importance for undergraduates has been left off or compromised by our approach. The payoff of the constructive approach, however, is that it makes sense—not just to math majors, but to students from all branches of the sciences.

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ACKNOWLEDGMENTS

About fifteen years ago, Gabe Stolzenberg lent me a copy of notes he had created and used to teach undergraduate seminars and directed studies courses in real analysis. I found Gabe's approach so elegant, and the material so appealing pedagogically, that I signed up to teach the analysis course, which had just become required for our majors. Over the years, with his help and encouragement, I worked this material into a text suited to the particular mix of students who take this course here at Northeastern. Many of the mathematical ideas in the current form of this text were adapted directly from Gabe's notes, especially the following: all of the material on the construction of the reals via rational interval arithmetic, the Inverse Function Theorem and its beautiful proof, the properties of exponential functions, and the creation of the Riemann integral. The use of the uniform notions of continuity and differentiability were also part of the "constructive mindset" that Gabe introduced me to, and which I have tried to employ throughout the remainder of the book. As a mathematician specializing in homological algebra, working with this new perspective was like writing a second thesis, with Gabe as advisor and friend.

Because Gabe's creative interests have taken him in other directions, this text could not be a joint authorship. He has continued to refine and expand his exposition of constructive mathematics, some of which can be found on his website. I am quite indebted to him—many of the good things in this book are due, directly or indirectly, to Gabe, and whatever is not so good is solely my responsibility.

Professor Joseph Alper read the entire manuscript and offered many extremely helpful suggestions and I am most indebted to him for this effort. Professor Robert Seeley very generously clarified a number of mathematical issues for me—especially those relating to Fourier analysis—as did my long-time office-mate Professor John Frampton. My wife, Maxine Bridger, not only proofread a lot of mathematics, she also kept me honest by countering my constructivist mindset with many a healthy platonist riposte.

Susanne Steitz of John Wiley & Sons skillfully shepherded this manuscript through the publishing process, and Anna Pierrehumbert did an extremely careful and insightful job of copyediting.

Finally, this entire manuscript was prepared using *Scientific Workplace*, a product of MacKichan Software. The technical support people there were extremely helpful, patient, and generous with their time.

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INTRODUCTION (MOSTLY FOR INSTRUCTORS)

Formally, Real Analysis—the course—is a presentation of the theoretical underpinnings of calculus. It is about the Big Three: continuity, differentiability and convergence. Yet it is also, for many, an introduction to reading, writing, and thinking mathematics. I have tried to address all of these issues in this book.

In the first chapter we construct the real numbers, starting with the rationals. This lays the groundwork for the entire book. The basic concept here is that of a family of rational intervals. A real number is a family of rational intervals which satisfies two important conditions: *consistency* (any two intervals in the family intersect) and *fineness* (the family contains intervals of arbitrarily small length). These conditions, together with the arithmetic that families inherit from the rationals, lead to all of the familiar algebraic properties of the reals. We establish these properties via quite a few propositions and one main theorem (completeness). Proving these results requires

- knowing simple properties of the arithmetic of rational numbers,
- applying elementary algebra and simple logic, and
- learning to apply new definitions and newly proved results.

Thus, Chapter 1 is critical because it provides not only the mathematical ideas that permeate the rest of the text, but also the introduction to the reasoning and writing skills necessary for doing and communicating mathematics. Nothing is more boring than having to read a seemingly endless theorem-proof sequence, so I have tried to provide just enough sample proofs and hints so that readers can proceed on their own. Many propositions are left as exercises; indeed, the exercises provide a vital part of the whole pedagogical process. I take this chapter at a leisurely pace, allowing students to write, critique, and rewrite their work. It is an investment of time well worth making early in the course.

Chapter 1 ends with what may be the central result of any real analysis course: the completeness of the reals. This is expressed in terms of families of real intervals, but in Chapter 3 it is rephrased in the language of Cauchy sequences.

Chapter 2 uses the Completeness Theorem to prove the useful Inverse Function Theorem. This, in turn, is used to construct nth roots, general exponential functions, and logarithms. A section is devoted to the Euler number e and the natural logarithm.

Chapter 3 introduces sequences, limits, and series and derives basic formulas and inequalities for the various functions already constructed.

In Chapter 4 we encounter uniform continuity. Since this version of continuity is the one most used in more advanced courses, we relegate the idea of pointwise continuity to the exercises. Nothing is lost, however, since the usual verifications of pointwise continuity for the basic functions of calculus are used with little modification to establish uniform continuity of these functions on intervals. We also encounter many interesting and important consequences of uniform continuity, among them boundedness and the extension of uniformly continuous functions from dense subsets—for example, extending functions from a punctured interval $[a, b] - \{x_1, \ldots, x_n\}$ to the closed interval $[a, b]$.

In Chapter 5, we use the Completeness Theorem again, this time to construct the Riemann integral $\int_a^b f$ for functions uniformly continuous on an interval $[a, b]$. The results previously established for limits and extensions of uniformly continuous functions can now be applied to define and calculate improper integrals. It is here that we introduce the important idea of functions defined as integrals. This includes the definition of the arctangent as an integral, an alternate definition of the natural logarithm (previously defined as an inverse function), and the use of improper integrals to construct the Gamma function and Laplace transforms.

Chapter 6 on differentiation emphasizes the derivative as a function rather than a pointwise limit. All the usual formulas from calculus are derived. In particular, the uniform version of differentiability that we use makes for very short and illuminating proofs of two central results of calculus:

- **The Law of Bounded Change**, which says that bounds for the derivative (i.e. $A \le f'(x) \le B$) are bounds for the difference quotient (i.e. $A \le \dfrac{f(y) - f(x)}{y - x} \le B$). (This is sometimes called the "Mean Value Inequality.")
- **The Fundamental Theorem of Calculus.**

In this chapter, we also derive some rather more difficult results on differentiating under the integral sign. In the case of improper integrals, we introduce "dominated convergence" assumptions, which we will also use later in studying series of functions.

In Chapter 7, nearly all of the ideas developed in the course are applied to studying the properties of sequences and series of continuous and differentiable functions. The particular case of power series is given special attention. The chapter ends with the definition of the periodic (trigonometric) functions as power series and a derivation of their properties (including a definition of π)—*all without pictures*. My students invariably enjoy this; in fact, with just a few simplifications and detours, it has even worked well for high school students taking AP calculus.

The last chapter of the book is organized around Fourier series, but it also provides an introduction to some of the more advanced ideas in functional analysis: inner products of functions, the Bessel and Cauchy-Schwartz inequalities and their applications, kernels and convolutions, and Abel summability. The early sections also introduce the complex numbers and the properties of complex-valued functions of a real variable.

There is enough material in the eight chapters to give a full-year course, especially if a lot of the more challenging exercises are assigned and discussed in class. Some of the exercises which have several parts and require more extensive work are labeled "projects."

I have usually given Real Analysis as a one-semester course. I generally get to cover the following.

1. Chapter 1: sections 1.0 through 1.7 (omitting 1.8 and skimming some of the material on absolute value and betweenness).
2. Chapter 2: in which I skip the more technical results—especially the 1- and 2-sided versions of the Inverse Function Theorem and some of the inequalities relating to the Euler number e.
3. Chapter 3: just what I need to talk about convergence of series.
4. Chapter 4: section 4.1 and the beginning of section 4.2 (omitting extensions of continuous functions), some material on limits from Chapter 3.
5. Chapter 5: sections 5.1 and 5.2.
6. Chapter 6: sections 6.1 through 6.3.
7. Chapter 7: just the material on power series.

Having done this for one semester, if there is enough student interest in a second semester, or a student wants to do a reading course, I can cover the more technical topics such as improper integrals, general convergence of sequences of functions, complex numbers, and Fourier series.

After teaching Real Analysis for many years, I'd say that my general experience has been that there is no general experience. Student ability, background and motivation can vary a lot from year to year, and I think it is a mistake to commit to a strict syllabus before you know your class. What is critical is that students do lots of problems and write lots of proofs. It is also very important that the central definitions and examples be memorized. I give several quizzes devoted exclusively to this. On the other hand, the more difficult material (proofs) is best tested via problem sets. Students seem to do these best—and enjoy them more—when working with one or two others. (But I do require independent write-ups!)

In terms of submitting mathematical work, most students initially write it out by hand. Since I typically require rewrites, many soon learn to use an equation editor with their word-processor. The software package *Scientific Notebook* is a good alternative, especially if you can get your school to underwrite its purchase. I have even had a few ambitious students learn to use TeX or LaTeX.

It is important to remember that this is an undergraduate course, and that most students taking it are probably not intending to go to graduate school in theoretical mathematics. The goal here is to have students understand the mathematics, be able to create some on their own, and come away with happy memories of the experience. There is also plenty of challenging material here, especially in the problems, for the talented and highly motivated student. The approach I have taken in this book has worked well over the years for me and my students. I hope it does for you and yours as well.

0. PRELIMINARIES

0.1 The Natural Numbers

You have to begin *somewhere*. We begin with the whole numbers: 0, 1, 2, ...and assume that we know what they are and that they have all the basic properties we know and love. Here are some of them:

1. (Commutative laws) $m+n=n+m$, $mn=nm$
2. (Associative laws) $k+(m+n)=(k+m)+n$, $k(mn)=(km)n$
3. (Distributive law) $k\cdot(m+n)=k\cdot m+k\cdot n$
4. (Additive identity) $m+0=m$
5. (Multiplicative identity) $m\cdot 1=m$
6. (Cancellation)
   (a) If $m+k=n+k$, then $m=n$.
   (b) If $m\cdot k=n\cdot k$ and $k\neq 0$, then $m=n$.
7. (Inequalities)
   (a) $m<n$ if and only if there is a non-zero whole number k with $m+k=n$.
   (b) $m\leq n$ if and only if $m>n$ is false.
   (c) For any k, $m+k<n+k$ if and only if $m<n$ (same for $\leq$).
   (d) For any $k\neq 0$, $m\cdot k<n\cdot k$ if and only if $m<n$ (same for $\leq$).
   (e) (Trichotomy) For any m, n, either $m<n$, $n<m$, or $m=n$.

There are, of course, many more such properties, but we will not attempt to list them all, nor will we try to prove any of them. Attempts have been made to derive the whole numbers and their properties solely from the "laws of logic" or from certain axioms for set theory, but we will not go down that path. In fact, it is not even clear that such an approach is worthwhile, since the existence and properties of whole numbers is arguably as basic and intuitive as the laws of logic or set theory (perhaps even more so).

We will denote the collection or set of whole numbers (including 0) by $\mathbb{N}$, standing for *natural numbers*.

NOTATION 0.1.1 *$n\in\mathbb{N}$ means that n is a natural number.*

One of the most useful properties of $\mathbb{N}$ is the following, which we have put in a box because of its importance.

Principle of Mathematical Induction

Suppose that S is a collection or set of natural numbers with the properties:
(a) S contains the number 0, and
(b) whenever S contains the number n it also contains $n+1$.
Then S is actually all of $\mathbb{N}$.

There are several alternative and equivalent versions of this principle; the version you use depends on the nature of the result you want to prove.

Variation 1: Suppose S contains k, and whenever S contains n it also contains $n+1$; then S contains all natural numbers $\geq k$.

Variation 2: Suppose S contains 0, and whenever S contains all the numbers from 0 through n it also contains $n+1$; then $S = \mathbb{N}$.

Mathematical induction is often compared to the behavior of dominos. The dominos are stood up on edge close to each other in a long row. When one is knocked over, it hits the next one (analogous to n in S implies $n+1$ in S), which in turn hits the next, etc. If then we hit the first (0 in S), then they will all eventually fall (S is all of $\mathbb{N}$). In Variation 1 above, we start by knocking over the kth domino, so that it and all subsequent ones eventually fall.

Here is an example of how a proof by induction works.

EXAMPLE 0.1.2 *Prove that for any $n \geq 1$, the sum of the first n odd numbers is n^2.*

Proof. We use Variation 1 above with $k = 1$. We first verify the claim when $n = 1$: the first odd number is 1 and the first square is $1^2 = 1$, so the claim holds in this case. Now we make the so-called "induction assumption" (or "induction hypothesis"), namely that the claim is true for some $n \geq 1$; so we have

$$1 + 3 + 5 + \cdots + \overbrace{(2n-1)}^{n\text{th odd number}} = n^2.$$

The idea is to use this to prove that the claim is true for the next number, $n+1$. So, starting with this equation, let's add the next odd number, $2n+1$, to both sides:

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1).$$

The left-hand side is the sum of the first $n+1$ odd numbers, while the right-hand side is, of course, equal to $(n+1)^2$. Thus, whenever the claim is true for a natural number $n \geq 1$ it is also true for $n+1$. All the dominos starting from $n = 1$ fall, and our proof is complete. ■

> WARNING: After stating the induction assumption you might be tempted to write $1+3+5+\cdots+(2n+1)=(n+1)^2$, in an attempt to display what it is that must now be proved. DON'T DO IT! You may be tempted to use it to prove itself. Always proceed *from what you know*, never from what you *want* to know. If you must work backwards, do it on scrap paper, but not in the final write-up.

A careful derivation of the arithmetic properties of the the natural numbers $\mathbb{N}$, using induction, was done by G. Peano (1858–1932). It is a lot of fun, but we will not pursue any of it here.

Once we have the natural number $\mathbb{N}$, it is a relatively easy and straightforward step to conceive of, or construct, the *integers* $\mathbb{Z}$ by adding on the "negatives" of the naturals. This gives us the collection consisting of $0, \pm 1, \pm 2, \ldots$ on which we have to define the laws of addition and multiplication, as well as the inequalities $<$ and $\leq$. There is a clever way of doing this which avoids dealing with a lot of the special cases that the straightforward approach entails. We sketch this in the exercises at the end of the chapter.

NOTATION 0.1.3 *$n \in \mathbb{Z}$ means that n is an integer.*

So, we can now turn to the fractions.

0.2 The Rationals

The natural numbers 1, 2, 3,... are used for counting discrete objects. In fact, the idea of counting is based on an assumption of discreteness. If a quantity to be measured is not a whole number of the units used to measure it, then the unit is subdivided into smaller units (yards into feet into inches into tenth inches etc.) and combinations of units are used (e.g. 2 feet 10 inches plus 3 tenth-inches). It is unclear when the idea that a single number could represent such a measurement was first thought of, although the Babylonians, with their uniquely advanced positional notation, may have achieved this realization. If one magnitude M_1 was used to measure another M_2 and the first didn't go into the second evenly, the school of Euclid referred to the relationship not as a number but as a ratio (this term itself being undefined). If a common unit could be chosen that measured each magnitude evenly, say $M_1 = a$ units and $M_2 = b$ units, then the ratio of magnitudes would be equal to a ratio of whole numbers, $M_1 : M_2 = a : b$. M_1 and M_2 were called *commensurable* in this case. Thus, a ratio of commensurables was basically an ordered pair of whole numbers, and various laws were given for dealing with them; for example, $ad : bd = a : b$. From Renaissance times this ordered pair $a : b$ has been denoted with a slash: a/b, and we now refer to it as a fraction or rational number instead of just a ratio. Historically, these ratios of whole numbers were accepted long before even negative whole numbers were used.

DEFINITION 0.2.1 (RATIONAL NUMBERS) *A rational number is an ordered pair a/b, where a and b are integers and $b \neq 0$.*

1. *$a/b = c/d$ means $ad = cb$.*

2. *$a/b < c/d$ means $\begin{cases} ad < cb & \text{if } bd > 0, \\ ad > cb & \text{if } bd < 0. \end{cases}$; this is also written $c/d > a/b$.*

In the definition above, we used the term "ordered pair." This can be defined in the context of set theory, but we will not pursue that formality. Basically, an ordered pair of integers—in this case denoted a/b but sometimes by (a, b)—is simply a pair of integers where the order makes a difference. In other words, a/b is not necessarily the same as b/a. In fact, in the definition we make an explicit condition for ordered pairs to be equal, namely $ad = cb$. This is an example of the power we have when we make a definition of something hitherto undefined. Rationals are added, subtracted, multiplied and divided in the usual way. It is not difficult to show that equal rationals produce equal results under these arithmetic operations; for example, $2/4 = 1/2$ and $2/4 + 3/7 = 1/2 + 3/7$.

Here is how the proof goes for addition. Suppose $a/b = a'/b'$ and $c/d = c'/d'$, so that $ab' = a'b$ and $cd' = c'd$. Now, using the familiar rule for adding fractions (which we take as our definition of rational number addition): $a/b + c/d = (ad + bc)/bd$ and $a'/b' + c'/d' = (a'd' + b'c')/b'd'$. Are the right-hand sides of these last two equations equal? Check that $(ad + bc)b'd' = (a'd' + b'c')bd$ by carefully multiplying out and using $ab' = a'b$ and $cd' = c'd$.

In the exercises we invite you to fill in the gaps in this quick exposition of the rational numbers. It is not essential to go through with all these details, but you might find it an enjoyable diversion, especially if you are not used to giving proofs in a non-geometric setting.

NOTATION 0.2.2 *$r \in \mathbb{Q}$ means that r is a rational number.*

The integers are located or represented or *embedded* in the rationals by the association: $a \to a/1$. When we refer to the rational integer a, for example, we will mean $a/1$.

A word about ordering: The integers are discrete in the sense that for any integer n there is no integer between n and $n + 1$. In other words, every integer has a "next" integer. This is not true of the rationals: for any two rationals $r = a/b$ and $s = c/d$, the rational $(r + s)/2 = \frac{1}{2}(ad + cb)/bd$, for example, lies strictly between r and s. In fact, there are infinitely many others in between r and s as well. However, even though the rationals are infinitely close together, given two rationals it is always possible to tell exactly which of the conditions $r < s$, $r = s$ or $r > s$ holds. This is because an order relation on rationals amounts to verifying an order relation between the integers ad and cb, and any two integers can be compared in a finite number of steps. The actual (practical) number of steps depends on how these integers are represented. In our base 10 positional notation, for example, two integers of roughly the same size n require at most about $\log_{10} n$ comparisons of the digits $0, \ldots, 9$, starting from the left. We summarize this computational fact in the following important result.

THEOREM 0.2.3 (TRICHOTOMY FOR THE RATIONALS) *For any two rational numbers r and s, exactly one of the following conditions can be determined: $r < s$, $r = s$, $r > s$.*

Suppose that we can rule out the case $r > s$. By Trichotomy we must have either $r < s$ or $r = s$. This defines what we might call the "weak" ordering of r and s, as opposed to the strong ordering $r < s$.

DEFINITION 0.2.4 *$r \leq s$ means that $r > s$ is false, i.e. $r \leq s$ means that either $r < s$ or $r = s$. This is also written $s \geq r$.*

COROLLARY 0.2.5 $a/b \leq c/d \iff \begin{cases} ad \leq cb & \text{if } bd > 0, \\ ad \geq cb & \text{if } bd < 0. \end{cases}$

Sometimes it is difficult to prove that $r \leq s$ but easier to prove that r is less than or equal to any number bigger than s. That this proves $r \leq s$ is often extremely useful, so we state it for rationals here, and will demonstrate it for real numbers later. We have called it the "Wiggle Lemma" because the presence of the ϵ gives us some "wiggle room," or leeway, in establishing the inequality.

LEMMA 0.2.6 (WIGGLE LEMMA FOR RATIONALS) *If r and s are rationals, and $r \leq s + \epsilon$ for every rational $\epsilon > 0$, then $r \leq s$.*

Proof. Let us assume $r \leq s + \epsilon$ for all rational $\epsilon > 0$; we will show that $r > s$ is impossible. If $r > s$, let $\epsilon = \dfrac{r-s}{2} > 0$. We have $r = \frac{1}{2}(r+r) > \frac{1}{2}(r+s) = s + \dfrac{r-s}{2} = s + \epsilon$, which contradicts our assumption. So $r > s$ is false. ■

REMARK 0.2.7 *Although the proof ends by establishing a contradiction, it is not an indirect proof. The statement $r \leq s$ is defined to mean "$r > s$" is false—i.e. produces a contradiction.*

The following special case occurs frequently in practice.

COROLLARY 0.2.8 *Suppose r and s are rationals and $r < s + \epsilon$ for every rational $\epsilon > 0$. Then $r \leq s$.*

Finally, we recall that $|r|$, the absolute value of r, is defined to be r if $r \geq 0$ and $-r$ otherwise. Absolute value has many interesting properties, many of which we will deal with later when we state them for real numbers. One of the most important is the following.

PROPOSITION 0.2.9 (TRIANGLE INEQUALITY FOR RATIONALS) $|a + b| \leq |a| + |b|$

(For a proof, see the exercises.)

Exercises

1. (Project) Here is an outline for constructing the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$. The idea is to let an integer be an ordered pair $\langle m, n\rangle$ of natural numbers m and n. This is a "formal" or abstract definition, but intuitively you should think of this as the integer $m - n$ (even though subtraction has not actually been defined for natural numbers). Since this is a definition of a new "object", we are free to define when these objects are to be equal. Intuitively, if $\langle m, n\rangle$ is $m-n$, and $\langle m', n'\rangle$ is $m'-n'$, then equality would give $m-n = m'-n'$. Since we have to build on just what we know for natural numbers, we write this equation as $m + n' = m' + n$ and make our definition:

   DEFINITION For integers, $\langle m, n\rangle = \langle m', n'\rangle$ means $m + n' = m' + n$.

   We can similarly define addition of integers: $\langle m, n\rangle + \langle m', n'\rangle = \langle m + n, m' + n'\rangle$ (since $(m - n) + (m' - n') = (m + m') - (n + n')$). Multiplication is a little trickier: $\langle m, n\rangle \cdot \langle m', n'\rangle = \langle mm' + nn', mn' + m'n\rangle$(can you see why?). Now, if $a = \langle m, n\rangle$, define $-a$ to be $\langle n, m\rangle$. Here are some things to prove:

   (a) The commutative and distributive laws for addition and multiplication.

   (b) If a, b and c are integers and $a = b$, then $a + c = b + c$

   (c) $a + (-a) = 0$.

   (d) $(-a)b = a(-b) = -ab$.

   (e) If $a + c = b + c$ then $a = b$ (hint: add $-c$ to both sides and use some of the results above).

   (f) $-(-a) = a$.

   You may also want to define the inequalities $\leq$ and $<$ and prove some of the properties for inequalities listed in the text for the natural numbers. If you do this, you can also define $|m|$ and prove that $|m|\,|n| = |mn|$.

   Note that the usual natural numbers can be considered integers by representing the natural number m by the integer $\langle m, 0\rangle$; thus, for example, 1 is represented by $\langle 1, 0\rangle$ (which, incidentally, is equal, as an integer, to $\langle 2, 1\rangle$ or $\langle 12, 11\rangle$, etc.). Now you can show that $1 \cdot a = a$, for example, by noting that $\langle 1, 0\rangle \cdot \langle m, n\rangle = \langle m, n\rangle$.

2. (Project) In the text we have defined rational numbers as ordered pairs a/b of integers ($b \neq 0$) with the equality relationship (for rationals) that $a/b = c/d$ means $ad = bc$. We also defined $a/b < c/d$ (see 0.2.1).

   DEFINITION For rationals $r = a/b$ and $s = c/d$:

   - (Sums and Negatives) $r + s = \dfrac{ad + bc}{bd}$, $-r = \dfrac{-a}{b} = \dfrac{a}{-b}$.
   - (Products and Reciprocals) $rs = \dfrac{ac}{bd}$, and if $r \neq 0$, then $1/r = \dfrac{b}{a}$.
   - (Differences and Quotients) $r - s = r + (-s)$, and if $s \neq 0$, then $r/s = r(1/s)$.

Here are some things to prove:

(a) The commutative and distributive laws for addition and multiplication.

(b) If r, s, and t are rationals, and $r = s$, then $rt = st$. (We proved a similar thing for addition of rationals in the text.)

(c) $r + (-r) = 0$, and if $r \neq 0$ then $r\,(1/r) = 1$.

(d) $(-r)s = r(-s) = -rs$.

(e) If $r + t = s + t$ then $r = s$. If $rt = st$ and $t \neq 0$, then $r = s$.

(f) $-(-r) = r$, and if $r \neq 0$, $1/(1/r) = r$.

(g) If $r > s$ and $t > 0$, then $rt > rs$. If $r < 0$, then $rt < rs$.

(h) Define $\leq$ for rationals and discuss its properties. Also, define $|r| = |m/n| = |m| \,/\, |n|$. Prove that $|rs| = |r|\,|s|$.

3. Prove the Triangle Inequality, $|a + b| \leq |a| + |b|$, for integers. The simplest way is to separate into the cases a and b both positive, both negative, and one positive, one negative. (You may assume that $m - n \leq m + n$ for natural numbers.)

4. Prove the Triangle Inequality $\left|\frac{a}{b} + \frac{c}{d}\right| \leq \left|\frac{a}{b}\right| + \left|\frac{c}{d}\right|$ for rationals by using the fact that it's true for integers (see previous exercise). You may assume that $|rs| = |r|\;|s|$ for rationals r and s. Hint: Consider multiplying or dividing through by $|bd|$.

The remaining exercises involve mathematical induction.

5. PROPOSITION If $n > 1$, then $n^2 > n + 1$.

Here are two proofs of this proposition; one is a good one and the other is bad form. Explain.

Proof #1. This is clearly true for $n = 2$. Suppose (induction hypothesis) that it's true for some number $n > 1$. We want to show $(n+1)^2 > (n+1) + 1$:

$$n^2 + 2n + 1 > n + 2$$

Now subtract $n + 1$ from both sides:

$$n^2 + n > 1.$$

This last statement is clearly true since n is at least 2. ∎

Proof #2. This is clearly true for $n = 2$. Suppose (induction hypothesis) that it's true for some number $n > 1$. We then have

$$n^2 > n + 1.$$

Since we want an $(n+1)^2$, add $2n+1$ to both sides:

$$n^2 + 2n + 1 > 3n + 2.$$

Since $n > 0$, $2n > 0$ so, adding $n+2$ to both sides yields $3n+2 > n+2$. Combining this with the above gives

$$(n+1)^2 = n^2 + 2n + 1 > 3n + 2 > n + 2 = (n+1) + 1.$$

Thus, the statement is true for $n+1$ and we are done. ■

6. Prove:

   (a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ when $n > 0$.

   (b) If $n \geq 4$, $n! > 2^n$.

   (c) $(1+x)^n \geq 1 + nx$. (You must make an assumption about x. What is it?)

   (d) For $0 \leq u \leq 1$ and integers $n > 0$, $(1-u)^n \leq 1 - u^n$. What about $n = 0$?

   (e) For $0 \leq u$ and integers $n > 0$, $1 - u^n \leq n(1-u)$.

   (f) If $n > 4$, $2^n > n^2$. (By the way, will 100^n eventually be bigger than n^{100}?)

7. Prove that $x^n - y^n$ is divisible by $x - y$ (as polynomials) (Hint: When looking at $x^{k+1} - y^{k+1}$, subtract and add yx^k in the middle.) Prove that $x^n + y^n$ is divisible by $x + y$ when n is odd.

8. The *Fibonacci sequence* $f_1, f_2, f_3 \ldots$ is defined as follows:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3.$$

   Prove the amazing formula:

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

   (Hint: Let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Then it is helpful to note that $\alpha + 1 = \alpha^2$ and $\beta + 1 = \beta^2$. Also, use Variation 2 of mathematical induction; i.e. in the induction step, show that if the claim is true for all numbers less than n, it is true for n as well.)

9. In studying the *Mandelbrot set*, one looks at a certain sequence of numbers $Z_0, Z_1, Z_2, \ldots$ which have the following properties:

$$\begin{aligned} Z_0 &= 0 \\ |Z_1| &= 2 + \varepsilon \text{ for a certain } \varepsilon > 0 \\ |Z_{k+1}| &\geq |Z_k|^2 - |Z_1| \text{ for all } k \geq 0. \end{aligned}$$

   Prove that, for all $N \geq 0$, $|Z_{N+1}| \geq 2 + N \cdot \epsilon$.

10. Prove by induction that the product of n consecutive integers is divisible by $n!$. (Use "double induction" to prove that $k(k+1)\ldots(k+n-1)$ is divisible by $n!$ by induction on n and k. This is a tricky problem and you must proceed very carefully!)

11. Let's make the ridiculous claim that any n things are equal. Here is a proof by induction. When $n = 1$, then it is clearly true, since any thing is equal to itself. Now suppose that it's true for some n and consider any $n+1$ things $x_1, x_2, \ldots x_n, x_{n+1}$. The first n of these are equal by induction assumption, and so are the last n:

$$\begin{array}{l} \overbrace{x_1,\ x_2,\ \ldots,\ x_n}^{\text{equal by induction}} \\ x_1,\ x_2,\ \ldots,\ x_n,\ x_{n+1} \\ \quad\ \underbrace{x_2,\ \ldots,\ x_n,\ x_{n+1}}_{\text{equal by induction}} \end{array}$$

Because of the overlap on the $n-1$ things $x_2, \ldots x_n$ we have $x_1 = x_2 = \cdots = x_n = x_{n+1}$, so $n+1$ things are always equal as well. What's wrong?

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1. THE REAL NUMBERS AND COMPLETENESS

1.0 Introduction

The real numbers are much more complicated than the integers or the rationals. A popular representation presents them as "infinite" decimals. Those infinite decimals that eventually repeat represent the rationals. Thus, for example, $\frac{3787}{9900}$ is represented as $0.3825252525\ldots = 0.38\overline{25}$,where the bar over the 25 indicate that it repeats. On the other hand, the infinite decimal representing $\sqrt{2}$, $1.414213562\ldots$, never repeats. Although we can define the collection of reals in this way, there are problems, the biggest one being how to define the arithmetic operations. For example, in order to add two infinite decimals, we have to "start" somewhere—at some decimal place "on the right." But then we have the problem of carries: some addition of digits to the right of where we start may require a carry that could effect the digits at or to the left of where we start. There is no simple way of dealing with this difficulty: any solution involves more and more complications. Furthermore, we lose the simplicity of rationals since even simple ones such as $3/17$ involve complicated, repeating, infinite decimals.

There have been several classical constructions of the reals that avoid these problems, the most famous ones being *Dedekind Cuts* and *Cauchy Sequences*, named respectively for the mathematicians Richard Dedekind (1831 - 1916) and Augustine Cauchy (1789 - 1857). We will not discuss these constructions here, but will use a more modern one developed by Gabriel Stolzenberg, based on "interval arithmetic." There are several advantages to this approach. The first is that we deal exclusively with rational numbers and their arithmetic (so we avoid the difficulties inherent in the infinite decimals approach). Secondly, as we discuss below, intervals of rational numbers are very much like scientific measurements, which, because of the fallibility of our instruments, are actually ranges of possible values. Finally, the analogy between interval arithmetic and scientific measurement allows us to apply theoretical results from the former to produce applications to the latter in the study of errors and error propagation.

So we begin with the study of rational intervals: $[r, s]$, where $r \leq s$ are rational numbers. We define their arithmetic—i.e. how to add, subtract, multiply, and divide them (as well as a few other manipulations). We also define the length of $[r, s]$ to be $s - r$, providing a measure of the "accuracy" of a measurement represented by this interval. Having done this, we turn to collections or *families* of rational intervals (representing a sequence or set of measurements). We define similar arithmetic operations on these families and then define two important properties that a family of intervals may have. The first of these, *consistency*, stipulates that any two intervals in the family intersect. The second, *fineness*, demands that the family contain intervals of arbitrarily small length. While this is seldom realizable in the real world, such a family can arise mathematically by taking partial sums of approximating series or other convergent numerical procedures. In fact, we will eventually define convergence using this idea.

Our principle definition is that a real number is a fine and consistent family of rational intervals. We will show that such families inherit the standard arithmetic operations (addition, multiplication, absolute value, etc.) from the rationals; indeed, we will prove all the usual properties of real numbers necessary to deal with algebraic equations and inequalities. Finally, we end by proving the central property that distinguishes the reals from the rationals: *completeness*. This takes the form of proving that any fine and consistent family of *real* intervals contains a unique real number common to all the intervals. Applications of completeness will be found throughout the following chapters, especially in the constructions of nth roots, exponential functions and logs, limits of series and functions, and the Riemann integral.

1.1 Interval Arithmetic

When a quantity—say a charge or weight, or volume—is measured, the instruments and methods used generally have a known or estimable accuracy. A numerical reading, say 5.647, usually comes with an error estimate, say ± 0.005. Thus, the quantity being measured lies in the range 5.647 ± 0.005, so is bounded below by 5.642 and above by 5.652. We can represent this by the interval $[5.642, 5.652]$. Since instruments read out either digitally or by dials and scales, the possible measurements resulting from this experiment are rational numbers lying in this interval. We now make some formal definitions related to this idea.

DEFINITION 1.1.1 *I is a rational interval means $I = [r, s]$, where r and s are rational numbers and $r \leq s$. The intervals $[r, s]$ and $[u, v]$ are equal precisely when $r = u$ and $s = v$.*

DEFINITION 1.1.2 *$x \in [r, s]$ means that x is a rational number and $r \leq x \leq s$.*

The rational numbers r and s are called the *endpoints* of the interval $[r, s]$. Please note that these definitions are fairly abstract. An interval is not defined to be a set but simply a symbol $[r, s]$. Unlike set theory, we *define* the membership relation $\in$. If you don't feel comfortable at this point with this degree of abstraction, that's ok: no harm will result if you think of the interval $[r, s]$ as simply the set of rationals bounded by the endpoints r and s.

EXAMPLE 1.1.3 $[2/5, 1/2] = [6/15, 2/4]$ *and* $7/15 \in [2/5, 1/2]$,*since* $2/5 \leq 7/15 \leq 1/2$.

We will often use the following notation: $I = [r, s]$; $J = [u, v]$; $K = [p, q]$ where $r \leq s$, $u \leq v$,and $p \leq q$ are rational numbers.

DEFINITION 1.1.4 (INCLUSION OF INTERVALS) $[r, s] \subset [u, v]$ *(or* $J \supset I$*) means that* $u \leq r$ *and* $s \leq v$.

Note that, since intervals are determined solely in terms of their endpoints, this property has been defined solely in terms of endpoints. It's usually a good idea to draw a picture to see intuitively what some condition means, so here is the picture for inclusion of intervals:

Here is a model for how proofs should go. It is a bit more wordy than it has to be, but that's ok for now.

PROPOSITION 1.1.5 $I \subset J$ *and* $J \subset K$ *implies that* $I \subset K$. *(More concisely:* $I \subset J$ *and* $J \subset K \Longrightarrow I \subset K$*).*

Proof. First we write the intervals in term of their endpoints: $I = [r, s], J = [u, v]$, and $K = [p, q]$. Now we go back to Definition 1.1.4 (and the above diagram) to unravel what our assumptions are saying:

$$\begin{aligned} I &\subset J \text{ means } u \leq r \text{ and } s \leq v \\ J &\subset K \text{ means } p \leq u \text{ and } v \leq q. \end{aligned}$$

We must apply the definition now to compare I and K using their endpoints. We can write $p \leq u \leq r$ and $s \leq v \leq q$. We conclude from these that $p \leq r$ and $s \leq q$ which, according to the definition, means $I \subset K$. This is what we wanted to prove. ■

The next two propositions assert that three statements "are equivalent." This means that when *any one* of them is true, so are *each of the others*. Thus, if the three statements are (1), (2), and (3), then we are actually making *six* assertions: (1) $\Longrightarrow$ (2), (1) $\Longrightarrow$ (3), (2) $\Longrightarrow$ (1), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2). However, you don't have to prove six separate implications, because it suffices to prove only (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), and (3) $\Longrightarrow$ (1). If we know these three, then, for example, (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1), so we get (2) $\Longrightarrow$ (1). The full proofs are left as exercises, though we include a piece of one to show what is expected.

PROPOSITION 1.1.6 *The following are equivalent:*

1. $I \subset J$
2. $x \in I \Longrightarrow x \in J$

3. J contains the endpoints of I.

Proof. We will prove just (1) $\Longrightarrow$ (2). Suppose, then, that (1) holds, so we are given $I \subset J$. With our usual notation, this means that $u \leq r$ and $s \leq v$ (see diagram above also). We must use this to prove $x \in I \Longrightarrow x \in J$, so assume that $x \in I$. By the definition of $\in$ above, this means that $r \leq x \leq s$. But, combining this with $u \leq r$ and $s \leq v$, we can write: $u \leq r \leq x \leq s \leq v$, so $u \leq x \leq v$, which means that $x \in J$. ■

PROPOSITION 1.1.7 *The following are equivalent:*

1. $I = J$
2. $x \in I \Longleftrightarrow x \in J$
3. $I \subset J$ *and* $J \subset I$

We now give the definitions of the fundamental arithmetic operations on intervals, also in terms of endpoints. Suppose, then, that $I = [r, s]$ and $J = [u, v]$.

DEFINITION 1.1.8 (ADDITION OF INTERVALS) $I + J$ *is the interval* $[r + u, s + v]$.

DEFINITION 1.1.9 (NEGATION OF INTERVALS) $-I$ *is the interval* $[-s, -r]$.

DEFINITION 1.1.10 (SUBTRACTION OF INTERVALS) $J - I$ *is the interval* $J + (-I) = [u - s, v - r]$.

NOTATION 1.1.11 (THE 0 INTERVAL) *We will occasionally denote the interval* $[0, 0]$ *simply by* 0.

Note that $I - I = [r, s] + [-s, -r] = [r - s, s - r]$ which is not in general equal to the interval 0; however, we do have the following result.

PROPOSITION 1.1.12 *For all intervals* I, $0 \in I - I$.

(Question: for which intervals I is it true that $I - I = [0, 0]$?)

DEFINITION 1.1.13 (MULTIPLICATION OF INTERVALS) *For* $I = [r, s]$ *and* $J = [u, v]$,

$$IJ = [\min(ru, rv, su, sv), \max(ru, rv, su, sv)].$$

When the intervals are "non-negative," their product is much simpler:

$$\text{if } a \geq 0, \text{ and } c \geq 0 \text{ then } [a, b][c, d] = [ac, bd].$$

However, with negative endpoints the situation is more complicated. For example,

$$[-3, 2][4, 5] = [-15, 10].$$

This definition of the product of two intervals, as you might expect, turns out to be rather difficult to apply when computing successive multiplications of intervals.