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HOCHSTADT

Integral Equations

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INTEGRAL EQUATIONS

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PREFACE

After having taught the subject of integral equations for a number of years I concluded that most potential texts are either too abstract, too old-fashioned, or too specialized. The classical techniques yield rather precise information regarding solvability of equations and existence and distribution of eigenvalues, but often require rather tedious and wearisome analysis. The functional analytic techniques yield certain kinds of results quite expeditiously, but often do not lead to the quantitative results provided by the classical techniques. The aim of this book is to compromise between these approaches to develop the most desirable features of each. The functional analytic approach can be developed in a Banach space setting, but to keep the treatment on a simpler level I have decided to restrict myself to Hilbert spaces.

Chapter 1 provides a heuristic discussion of integral equations based on finite difference approximations. It is assumed that the reader has some background in linear algebra, but the more specialized results of that subject that are needed later are developed here. There is also a discussion of the necessary background of Hilbert space theory.

Chapter 2 develops some elementary techniques, in particular the contraction mapping principle and its applications to integral equations.

Chapter 3 develops the theory of compact operators, which is then used to discuss a broad class of integral equations. Another long section is devoted to ordinary differential operators and their study via compact integral operators. The Fredholm alternative is discussed fully.

In Chapter 4 some applications of these techniques to boundary value problems in more than one dimension are discussed.

Chapter 5 is devoted to a complete treatment of numerous transform techniques. The Fourier transform is developed and used to develop the Laplace, Mellin, and Hankel transforms. Subsequently, the projection method is discussed and applied to Wiener-Hopf problems and to certain mixed boundary value problems. Some of the ad hoc methods for these problems are more direct than the method presented here. It is my feeling, however, that it is most desirable to discuss these problems in the framework of a unified and consistent theory. All too often the student is left with the idea that such problems are solved by tricks rather than within the framework of a consistent theory.

Chapter 6 develops the classical Fredholm technique to obtain a number of results that the functional analytic techniques do not yield. Integral operators with positive kernels are also discussed.

Chapter 7 is a distinct departure from the preceding chapters. Almost all the earlier chapters are devoted to linear integral equations. The theory of nonlinear equations is not nearly as well developed as the linear theory. Here the Schauder fixed-point theorem is presented and applied to a number of nonlinear equations. A complete proof of the Schauder theorem is given, based on the Brouwer fixed-point theorem. The latter is proved by analytic rather than topological methods.

Without a doubt there are many topics that might well have been included with profit. Notable examples can be cited of such omissions. One of the most powerful and beautiful applications of the Fredholm alternative can be found in potential theory. Chapter 3 could well have been enriched by the inclusion of this topic. Because this topic is so tempting to an author, I decided to exclude it; it can be found in many other works on integral equations. Systems of integral equations, generalizations from Hilbert to Banach spaces, more general Wiener-Hopf equations, all could have been presented. No book can pretend to be complete and exhaustive. Every author has to decide at what point he is willing to stop for economic and pedagogical reasons. Experience has shown that the following material can be covered in a one-year course. Hopefully, the mixture is such that both applied scientists and mathematicians, to whom I intended to address myself, will find this material useful and interesting for its own sake.

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1

GENERAL INTRODUCTION

SUMMARY

This chapter is devoted to a discussion of some of the broad classes of integral equations to be discussed in more detail in later chapters. By means of examples and finite difference approximations, some motivation will be given for the differences in theory and methodology underlying these classifications and their investigation.

The necessary background in linear algebra will be sketched and some aspects of Hilbert space theory will be presented. Subsequently, we will view all integral operators to be discussed as operators acting on suitable Hilbert spaces.

1. INTRODUCTION

The theory of integral equations has close contacts with many different areas of mathematics. Foremost among these are differential equations and operator theory. Many problems in the fields of ordinary and partial differential equations can be recast as integral equations. Many existence and uniqueness results can then be derived from the corresponding results from integral equations.

Many problems of mathematical physics can be stated in the form of integral equations. Some of these will be discussed as examples and treated explicitly. To make a list of such applications would be almost impossible. Suffice it to say that there is almost no area of applied mathematics and mathematical physics where integral equations do not play a role.

In many ways one can view the subject of integral equations as an extension of linear algebra and a precursor of modern functional analysis. Especially in dealing with linear integral equations the fundamental concepts of linear vector spaces, eigenvalues and eigenfunctions will play a significant role.

There are a number of classifications of linear integral equations that distinguish different kinds of equations. The following are the most frequently studied.

$$\int_a^b K(x, y)\phi(y)\,dy = f(x), \tag{1}$$

$$\phi(x) - \lambda\int_a^b K(x, y)\phi(y)\,dy = f(x), \tag{2}$$

$$a(x)\phi(x) - \lambda\int_a^b K(x, y)\phi(y)\,dy = f(x). \tag{3}$$

The above equations (1)–(3) are generally known as Fredholm equations of the first, second, and third kind, respectively. The interval (a, b) may in general be a finite interval or $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$, where a and b are finite. If $a(x)$ does not vanish one can divide (3) by $a(x)$ to reduce it to (2). The functions $f(x)$, $a(x)$, and $K(x, y)$ are presumably known functions and the function $\phi(x)$ is unknown. The parameter λ could be absorbed in the function $K(x, y)$, but it is convenient to retain it in the equation. Its role will become clearer when the operators in question will be studied. The function $K(x, y)$ is generally known as the kernel of the equation.

A second class of equations are the Volterra equations of the first, second, and third kind, namely

$$\int_a^x K(x, y)\phi(y)\,dy = f(x), \tag{4}$$

$$\phi(x) - \lambda\int_a^x K(x, y)\phi(y)\,dy = f(x), \tag{5}$$

$$a(x)\phi(x) - \lambda\int_a^x K(x, y)\phi(y)\,dy = f(x). \tag{6}$$

One can view these as special cases of Fredholm equations. The latter reduce to the corresponding Volterra equations if $K(x, y) = 0$ for $y > x$. Nevertheless Volterra equations have many interesting properties that do not emerge from the general theory of Fredholm equations so that a separate study is definitely warranted.

Equations (1)–(6) have one thing in common; they are all linear equations. That is, the function ϕ enters the equations in a linear manner so that

$$\int_a^b K(x, y)[c_1\phi_1(y) + c_2\phi_2(y)]\,dy = c_1\int_a^b K(x, y)\phi_1(y)\,dy + c_2\int_a^b K(x, y)\phi_2(y)\,dy.$$

If the integral were replaced by the more general

$$\int_a^b K(x, y, \phi(y))\, dy$$

one would call the equation nonlinear. Typical examples of such operators are

$$\int_a^b K(x, y)\phi^2(y)\, dy,$$

$$\int_a^b K(x, y) \sin \phi(y)\, dy.$$

2. EXAMPLES

Consider the Volterra equation

$$\phi(x) - \lambda \int_0^x y\phi(y)\, dy = f(x). \tag{7}$$

By differentiation, the above can be reduced to a first order differential equation

$$\phi'(x) - \lambda x\phi(x) = f'(x),$$

and from (7) we obtain the initial value

$$\phi(0) = f(0).$$

One can solve this equation by standard methods of differential equations to obtain

$$\phi(x) = f(x) + \lambda \int_0^x e^{\lambda/2(x^2-y^2)} f(y)\, dy. \tag{8}$$

One can easily check that (8) satisfies the differential equation as well as the integral equation (7). To derive the differential equation we required that both $\phi(x)$ and $f(x)$ had first derivatives. The differentiation was merely an artifice that enabled us to reduce the integral equation to a differential equation, whose solution was obtainable by relatively elementary methods.

As a second example we consider the Fredholm equation

$$\phi(x) - \lambda \int_0^1 xy(y - x)\phi(y)\, dy = f(x). \tag{9}$$

The expressions $\lambda \int_0^1 y^2\phi(y)\, dy$ and $-\lambda \int_0^1 y\phi(y)\, dy$ are unknown constants depending on $\phi(y)$ and we shall denote them by a and b. Then (9) can be rewritten in the form

$$\phi(x) = f(x) + ax + bx^2. \tag{10}$$

The fact that the solution of (9) has the simple character of (10) depends on the special nature of the kernel in (9). To determine a and b we note that by multiplying (10) by λx^2 and integrating

$$\lambda\int_0^1 x^2\phi(x)\,dx = \lambda\int_0^1 x^2 f(x)\,dx + \lambda\frac{a}{4} + \lambda\frac{b}{5} = a.$$

Similarly,

$$\lambda\int_0^1 x\phi(x)\,dx = \lambda\int_0^1 xf(x)\,dx + \lambda\frac{a}{3} + \lambda\frac{b}{4} = -b.$$

The latter are two linear algebraic equations in two unknowns a and b. These can be solved by standard methods, and inserting the solutions in (10) one finally obtains

$$\phi(x) = f(x) + \lambda\int_0^1 \frac{xy\{-(\lambda/5) + [1+(\lambda/4)]y - (\lambda/3)xy - [1-(\lambda/4)]x\}f(y)}{1+(\lambda^2/240)}\,dy \tag{11}$$

The two examples (7) and (9) and the corresponding solutions (8) and (11) demonstrate one of the big differences between Volterra and Fredholm equations. The solution (8) exists for all finite values of λ, whereas (11) will fail to exist if its denominator $1 + (\lambda^2/240)$ vanishes. In that case (11) will in general fail to have a solution. If, however,

$$\int_0^1 x^2 f(x)\,dx = 0, \qquad \int_0^1 xf(x)\,dx = 0, \qquad 1 + \frac{\lambda^2}{240} = 0$$

(9) will have the solution

$$\phi(x) = f(x) + \frac{\lambda}{5}cx - \left(1 - \frac{\lambda}{4}\right)cx^2 \tag{12}$$

where c is completely arbitrary.

We saw that (7) could be solved by reducing it to a differential equation. In fact, the first order differential equation

$$\phi'(x) = f(x, \phi(x)) \qquad \phi(0) = \phi_0 \tag{13}$$

can be rewritten in the form of a nonlinear Volterra equation, by integrating (13).

$$\phi(x) = \phi_0 + \int_0^x f(y, \phi(y))\,dy. \tag{14}$$

In fact, the proof of the existence and uniqueness of a solution for (13) is generally based on treating the integral equation (14).

Fredholm equations also can be related to certain types of differential equations. Consider the boundary value problem

$$\phi''(x) + \lambda\phi(x) = f(x) \qquad \phi(0) = 0 \qquad \phi(1) = 0. \tag{15}$$

A double integration of (15) shows that

$$\phi(x) = c_1 + c_2 x + \int_0^x (x - y)[f(y) - \lambda\phi(y)]\,dy.$$

From $\phi(0) = 0$ we conclude that $c_1 = 0$. From $\phi(1) = 0$ we see that

$$c_2 = -\int_0^1 (1 - y)[f(y) - \lambda\phi(y)]\,dy.$$

Use of these values of c_1 and c_2 allows us to rewrite problem (15) in the form

$$\phi(x) - \lambda\int_0^1 K(x, y)\phi(y)\,dy = -\int_0^1 K(x, y)f(y)\,dy \tag{16}$$

where

$$\begin{aligned} K(x, y) &= y(1 - x), \qquad y \le x \\ &= x(1 - y), \qquad y \ge x. \end{aligned}$$

Equation (16) is equivalent to (15) and incorporates the boundary conditions as well.

Volterra equations of the first kind are in some ways more difficult than equations of the second kind. For example, the apparently simple equation

$$\int_0^x y\phi(y)\,dy = f(x) \tag{17}$$

has the solution

$$\phi(x) = \frac{f'(x)}{x}. \tag{18}$$

But this solution can make sense only if $f(x)$ satisfies certain regularity conditions. From (17) we see that $f(0) = 0$. Furthermore $f(x)$ must be differentiable for (18) to make sense. In (7) we used the differentiability of $f(x)$ merely as an artifice that could be dispensed with in (8). In (18) the differentiability is vital.

3. FINITE DIFFERENCE APPROXIMATIONS

Finding explicit solutions of integral equations is in general just as difficult as finding solutions of differential equations. Only in exceptional cases can such solutions be found. Generally, various approximate and numerical methods have to be used. Finite difference approximations are not only of great practical utility, but also provide certain insight into the nature of integral equations.

In the equation

$$\phi(x) - \lambda\int_0^1 K(x, y)\phi(y)\,dy = f(x) \tag{19}$$

we shall replace the integral by a suitable sum:

$$\phi(x) - \lambda \sum_{i=1}^{n} \frac{1}{n} K\left(x, \frac{i}{n}\right) \phi\left(\frac{i}{n}\right) = f(x). \tag{20}$$

For large n and a continuous kernel $K(x, y)$ and continuous $\phi(x)$ the sum in (20) represents a close approximation to the integral in (19). If, furthermore, we evaluate (20) only at n discrete points

$$\phi\left(\frac{j}{n}\right) - \lambda \sum_{i=1}^{n} \frac{1}{n} K\left(\frac{j}{n}, \frac{i}{n}\right) \phi\left(\frac{i}{n}\right) = f\left(\frac{j}{n}\right), \qquad j = 1, 2, \ldots, n, \tag{21}$$

we have replaced the integral equation (19) by the algebraic system (21). We shall rewrite (21) in matrix form

$$(I - \lambda L)\Phi = F \tag{22}$$

where the general term in the matrix L is $(1/n)K(j/n, i/n)$, and Φ is a vector with components $\phi(i/n)$ and F has components $f(i/n)$.

To solve (22) we invert the matrix and find

$$\Phi = (I - \lambda L)^{-1}F. \tag{23}$$

The above inverse will exist for all λ, with the exception of at most n values. These are the roots of the characteristic determinantal equation

$$|I - \lambda L| = 0. \tag{24}$$

In (23) as in (11) we see that there may be special values of λ for which no solution exists. Such values are commonly known as eigenvalues. Every finite algebraic system (21) necessarily has eigenvalues, even though the integral equation (19) need not have eigenvalues. This is a subtle point that we shall bypass for the time being.

For the case where (19) is a Volterra equation $K(x, y) = 0$ for $y \geq x$. In that case (21) can be rewritten as

$$\phi\left(\frac{j}{n}\right) - \lambda \sum_{i=1}^{j-1} \frac{1}{n} K\left(\frac{j}{n}, \frac{i}{n}\right) \phi\left(\frac{i}{n}\right) = f\left(\frac{j}{n}\right), \qquad j = 1, 2, \ldots, n. \tag{25}$$

In this case L in (22) is in fact a triangular matrix with all diagonal elements equal to zero:

$$L = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ l_{21} & 0 & 0 & \cdots & 0 & 0 \\ l_{31} & l_{32} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{n,n-1} & 0 \end{pmatrix} \tag{26}$$

Inspection of (26) shows that L is a nilpotent matrix and in particular $L^n = 0$. This fact enables us to find the inverse of $I - \lambda L$ for all λ. A simple calculation shows that

$$(I - \lambda L)(I + \lambda L + \lambda^2 L^2 + \cdots + \lambda^{n-1} L^{n-1}) = I \tag{27}$$

so that (23) can be replaced by

$$\Phi = (I + \lambda L + \lambda^2 L^2 + \cdots + \lambda^{n-1} L^{n-1}) F. \tag{28}$$

As in (8), (28) is another heuristic demonstration that Volterra equations of the second kind will have unique solutions for all λ.

Equations of the first kind are in general more complicated. If we study the finite difference analog of a Fredholm equation of the first kind we are led to a system of the type

$$L\Phi = F. \tag{29}$$

Now if $|L| \neq 0$, L has an inverse and (29) has a unique solution. If $|L| = 0$, (29) may or may not have a solution. But even if a solution exists it will not be unique.

For the case where (29) represents a Volterra equation, L is nilpotent, as in (26) and necessarily $|L| = 0$, so that the above remarks apply.

4. THE FREDHOLM ALTERNATIVE

The Fredholm alternative is a fundamental tool in reaching decisions regarding the solvability of certain types of integral equations. In this section the analogous theory for finite dimensional spaces will be sketched. Let X be a finite dimensional inner product space of dimension n over the field of complex numbers and L an operator on X that is represented by an $n \times n$ matrix. With L we can associate the adjoint matrix L^*, which is sometimes referred to as the hermitean transpose of L.

Let (ϕ, ψ) denote the inner product defined on X, where ϕ and ψ are arbitrary vectors in X. L and L^* are related by the expression

$$(L\phi, \psi) = (\phi, L^*\psi).$$

The above can in fact be used as the defining relationship for L^*.

By $N(L)$ we denote the nullspace of L, that is

$$N(L) = \{\phi \mid L\phi = 0\}.$$

It is easy to verify that $N(L)$ is indeed a subspace of X. By $\nu(L)$ we shall denote the dimension of $N(L)$. By $R(L)$ we shall denote the range of L, so that

$$R(L) = \{\phi \mid \phi = L\psi \text{ for some } \psi \in X\}$$

Again we can see that $R(L)$ is a subspace and its dimension is denoted by $\rho(L)$.

LEMMA 1. $\nu(L) + \rho(L) \leq n = \dim X$.

PROOF. Let $\phi_1, \phi_2, \ldots, \phi_n$ be a basis for X such that $\phi_1, \phi_2, \ldots, \phi_\nu$ is a basis for $N(L)$. Then for any $\phi \in X$ we have

$$\phi = \sum_{i=1}^{n} \alpha_i \phi_i$$

for a suitable set of scalars α_i. It follows that

$$L\phi = \sum_{i=\nu+1}^{n} \alpha_i L\phi_i$$

since $L\phi_i = 0$ for $1 \leq i \leq \nu(L)$. Since $L\phi$ is a general element in $R(L)$ and since it is a linear combination of at most $n - \nu(L)$ linearly independent vectors we see that

$$\rho(L) \leq n - \nu(L)$$

thereby proving the lemma. ∎

It may be remarked that the above inequality is in fact an equality. This will be proved shortly.

If R is a subspace of X we note the set of all vectors in X that are orthogonal to R by $R^\perp$. $R^\perp$ can be shown to be a subspace of X and is called the orthogonal complement of R.

LEMMA 2. $\nu(L) + \rho(L^*) = n$.

PROOF. By $R(L^*)^\perp$ we denote the orthogonal complement of $R(L^*)$. We shall first prove that $N(L) \subset R(L^*)^\perp$. Let $\phi \in N(L)$ so that

$$0 = (L\phi, \psi) = (\phi, L^*\psi) \qquad \text{for all } \psi.$$

It follows that ϕ is orthogonal to all $L^*\psi$ so that

$$\phi \in R(L^*)^\perp.$$

Since ϕ is an arbitrary element of $N(L)$ we see that

$$N(L) \subset R(L^*)^\perp.$$

We now prove the reverse inclusion. Let $\phi \in R(L^*)^\perp$ so that

$$0 = (\phi, L^*\psi) = (L\phi, \psi) \qquad \text{for all } \psi.$$

From the above we see that necessarily $\phi \in N(L)$ so that

$$R(L^*)^\perp \subset N(L).$$

We can conclude that $N(L) = R(L^*)^{\perp}$. From this we conclude that $N(L)$ and $R(L^*)$ form complementary subspaces of X so that

$$X = N(L) \oplus R(L^*).$$

That is, every element of X can be uniquely decomposed into an element in $N(L)$ and an element of $R(L^*)$. It follows that

$$\nu(L) + \rho(L^*) = n. \quad \blacksquare$$

LEMMA 3. $\nu(L) = \nu(L^*)$.

PROOF. On a finite dimensional space the double adjoint $L^{**} = L$. By lemmas 1 and 2 we have

$$\nu(L^*) \leq n - \rho(L^*) = \nu(L).$$

Now we see that

$$\nu(L) = \nu(L^{**}) \leq \nu(L^*) \leq \nu(L)$$

so that $\nu(L) = \nu(L^*)$. $\blacksquare$

COROLLARY. $\nu(L) + \rho(L) = n$.

PROOF. By lemmas 2 and 3

$$n = \nu(L^*) + \rho(L) = \nu(L) + \rho(L). \quad \blacksquare$$

The above shows that the inequality in lemma 1 is in fact an equality. However, $N(L)$ and $R(L)$ are not, in general, complementary subspaces as are $N(L)$ and $R(L^*)$. In the case where L is selfadjoint so that $L = L^*$ they will be, of course.

THEOREM 1. The Fredholm Alternative. The equation

$$L\phi = f \tag{30}$$

has a unique solution for all f, if and only if $\nu(L) = 0$. If $\nu(L) > 0$ the above equation has solutions only for those f, that are orthogonal to the nullspace $N(L^*)$.

PROOF. If $\nu(L) = 0$ we see from the above corollary that $\rho(L) = n$ so that the range $R(L)$ is the entire space. In that case, f necessarily belongs to $R(L)$ and can be represented by $L\phi$ for some ϕ. If there were two such solutions ϕ_1 and ϕ_2, then $L(\phi_1 - \phi_2) = 0$. But since $\nu(L) = 0$, we see that $\phi_1 = \phi_2$.
If $\nu(L) > 0$, then by lemma 2, (30) has a solution only for those $f \in R(L) = N^{\perp}(L^*)$. Therefore solutions exist only for those f orthogonal to $N(L^*)$. $\blacksquare$

5. HADAMARD'S INEQUALITY

A familiar result from analytic geometry yields the volume of a parallelepiped in terms of a determinant. Let (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) denote the three vectors defining three edges of such a parallelepiped. Then the absolute value of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

furnishes the volume V of this parallelepiped. A simple upper estimate for such a volume is given by the following inequality:

$$V \leq \sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2)}. \tag{31}$$

Equality is attained if the three edges are mutually perpendicular. Under all other conditions (31) is a strict inequality.

THEOREM 2. Hadamard's Inequality. Let L be a matrix with the general element l_{ij}. An upper estimate for its determinant is given by

$$|L|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |l_{ij}|^2. \tag{32}$$

Equation (32) is a generalization of (31) in two ways. First of all it is a statement that applies to an n dimensional space. Here our geometric intuition no longer suffices. Secondly, we allow the terms l_{ij} to be complex numbers whereas in (31) we dealt only with real numbers. In order to prove (32) we require some preliminary results.

THEOREM 3. On the Arithmetic-Geometric Mean. Let $d_1, d_2, \ldots, d_n$ be a set of n non-negative numbers. Then

$$\frac{1}{n} \sum_{i=1}^{n} d_i \geq \prod_{i=1}^{n} d_i^{1/n}. \tag{33}$$

PROOF. LET A denote the arithmetic mean $(1/n) \sum_{i=1}^{n} d_i$. If all d_i are equal the above inequality reduces to an equality. More generally, let d_M and d_m denote the greatest and smallest of the d_i respectively. Define a and b by

$$d_M = A + a, \qquad d_m = A - b.$$

We now replace d_M by A and d_m by $A + a - b$. This substitution has evidently no effect on the arithmetic mean. But in view of the fact that

$$d_M d_m = A(A + a - b) - ab \leq A(A + a - b)$$

this substitution will increase the geometric mean $\prod_{i=1}^{n} d_i^{1/n}$. If a substitution of the above type is performed repeatedly, and each time the greatest of the d_i is replaced by A, we see that after at most $n - 1$ steps all d_i have been replaced by A. Each such step leaves the arithmetic mean unchanged, but increases the geometric mean. After $n - 1$ steps all d_i have been replaced by A so that the geometric mean and arithmetic mean are equal. It follows that in general (33) must be a strict inequality, unless all d_i are equal. ■

An $n \times n$ matrix L is said to be positive, if $(L\phi, \phi) \geq 0$, for all vectors ϕ in the inner product space X. For such matrices the following theorem yields an estimate that will enable us to prove the Hadamard inequality.

THEOREM 4. Let L be a selfadjoint, positive matrix. Then

$$|L| \leq \prod_{i=1}^{n} l_{ii}. \tag{34}$$

PROOF. The positivity of L implies that $(L\Phi, \Phi) \geq 0$ for all Φ. If $|L| = 0$ (34) is trivial. If $|L| \neq 0$, $(L\Phi, \Phi) > 0$ for all $\Phi \neq 0$. Now let Φ_i be the ith canonical unit vector, namely $\Phi_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$. Here δ_{ij} denotes the Kronecker δ defined by

$$\begin{aligned} \delta_{ij} &= 0 \qquad i \neq j \\ &= 1 \qquad i = j. \end{aligned}$$

Then

$$(L\Phi_i, \Phi_i) = l_{ii} > 0.$$

Let D denote the diagonal matrix with diagonal terms $1/\sqrt{l_{11}}, 1/\sqrt{l_{22}}, \ldots, 1/\sqrt{l_{nn}}$. Then we consider the matrix $T = DLD$. First of all T is selfadjoint since

$$T^* = D^*L^*D^* = DLD = T.$$

It is also positive, since

$$(T\Phi, \Phi) = (DLD\Phi, \Phi) = (LD\Phi, D\Phi) \geq 0.$$

It is also evident that the diagonal terms of T are all equal to 1. We also see that

$$|T| = |D|^2 |L| = |L| \prod_{i=1}^{n} \frac{1}{l_{ii}}. \tag{35}$$

We shall show that $|T| \leq 1$, so that (35) yields the desired result (34).

To establish the fact that $|T| \leq 1$, we denote the eigenvalues of T by λ_i. It follows that

$$\text{trace } T = n = \sum_{i=1}^{n} \lambda_i.$$

$$|T| = \prod_{i=1}^{n} \lambda_i.$$

We now have, by theorem 3

$$1 = \frac{1}{n}\sum_{i=1}^{n} \lambda_i \geq \left[\prod_{i=1}^{n} \lambda_i\right]^{1/n} = |T|^{1/n}$$

so that $|T| \leq 1$, which implies (34). ■

PROOF OF HADAMARD'S INEQUALITY. Let L be a general matrix. Then LL^* is clearly selfadjoint and positive, since

$$(LL^*\Phi, \Phi) = (L^*\Phi, L^*\Phi) = \|L^*\Phi\|^2 \geq 0.$$

The general term of LL^* is $\sum_{k=1}^{n} l_{ik}\bar{l}_{jk}$ and an application of (34) to LL^* yields

$$|LL^*| = |L|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |l_{ij}|^2. \quad ■$$

COROLLARY. Let $L = (l_{ij})$ and $|l_{ij}| \leq l$. Then

$$|L| \leq l^n n^{n/2}.$$

PROOF. Use of (34) shows that

$$|L|^2 \leq \prod_{i=1}^{n} nl^2 = n^n l^{2n}.$$

Taking square roots yields the result. ■

6. HILBERT SPACES

So far we have not really addressed ourselves to the question of what we mean by a solution of an integral equation. Basically, of course, a solution of any equation must reduce the equation to an identity. But often one may impose additional restrictions on the solution, such as demanding that it should belong to a particular class of functions. For these purposes it will prove to be convenient to work in so-called Hilbert spaces.

DEFINITION. A linear space over a field F (in our work F will invariably be the field of complex numbers) is a collection X of elements with two defined operations. The first of these is addition of elements in X and the second multiplication of elements in X by scalars in F. In addition we stipulate the following conditions:

1. X forms a commutative group under the additive operation. That is, if $f, g, h, \ldots$ belong to X, then

(a) the operation is closed so that $f + g$ belongs to X for all f and g in X.
(b) the operation is associative: $(f + g) + h = f + (g + h)$.

(c) there exists an identity 0 for which $f + 0 = f$ for all f in X.
(d) for every f there exists an inverse element denoted by $(-f)$ such that

$$f + (-f) = 0.$$

(e) the operation is commutative

$$f + g = g + f \qquad \text{for all } f, g \text{ in } X.$$

2. Multiplication by scalars is closed. That is,

(a) $1 \cdot f = f$ for all f in X,
(b) αf is in X for all f in X and α in F.
(c) for all α, β in F and f in X

$$\alpha(\beta f) = (\alpha\beta)f.$$

3. The following distributive laws hold:

(a) $\alpha(f + g) = \alpha f + \alpha g$ for all α in F and f, g in X.
(b) $(\alpha + \beta)f = \alpha f + \beta f$ for all α, β in F and f in X.

EXAMPLE 1. Consider the set of all vectors of the form $f = (a_1, a_2, \ldots, a_n, \ldots)$ where all a_i are complex numbers, and only a finite number of a_i do not vanish. If $g = (b_1, b_2, \ldots, b_n, \ldots)$ is a second such vector we define

$$\alpha f = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n, \ldots)$$

where α is any complex number, and

$$f + g = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots).$$

It is a simple matter to verify that the above set of vectors forms a linear space.

EXAMPLE 2. Consider the set of all continuous functions $f(x)$ defined on the closed interval $[0, 1]$, that take complex values. We denote this set by $C[0, 1]$. For any complex scalars $(\alpha f)(x) = \alpha \cdot f(x)$ in $C[0, 1]$ if $f(x)$ is in $C[0, 1]$. Addition is defined pointwise, that is

$$f + g = f(x) + g(x).$$

Again, one can verify that these form a linear space.

The linear spaces do not have enough structure to be useful in the area of integral equations. But the inner product spaces prove to be much more useful.

DEFINITION. A linear space X is said to be an inner product space if an inner product is defined on it. Such an inner product assigns to every pair

f and g in X a complex number denoted by (f, g). By definition, such an inner product has the following properties:

1. $(f, g) = \overline{(g, f)}$
2. $(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$
3. $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$.

N.B. (f, f) is real, since by property 1 $(f, f) = \overline{(f, f)}$. We let $(f, f)^{1/2} = \|f\|$ and call it the norm of f.

THEOREM 5. Cauchy-Schwarz Inequality. Let f and g belong to an inner product space. Then

$$|(f, g)| \leq \|f\| \, \|g\| \tag{35}$$

Equality is achieved if and only if f and g are linearly dependent; that is for suitable scalars α and β, $\alpha f + \beta g = 0$.

PROOF. We assume that $f \neq 0$, because if $f = 0$, (35) is trivially true. Let

$$(f, g) = |(f, g)| \, e^{i\theta}, \qquad \alpha = \frac{|(f, g)| \, e^{-i\theta}}{\|f\|^2}$$

so that

$$(\alpha f - g, \alpha f - g) \, \|f\|^2 = \|\alpha f - g\|^2 \, \|f\|^2 \geq 0.$$

Using the properties of inner products we can expand the left side of this inequality to obtain

$$[|\alpha|^2 \, \|f\|^2 - \alpha(f, g) - \overline{\alpha(f, g)} + \|g\|^2] \, \|f\|^2 \geq 0.$$

So far no use has been made of the particular value of α. If that value is inserted in the above inequality, we obtain

$$\|g\|^2 \, \|f\|^2 - |(f, g)|^2 \geq 0$$

which is equivalent to (35). To complete the proof we note that if f and g are linearly independent

$$\|\alpha f - g\| > 0$$

so that we get a strict inequality. Equality will occur if $g = \alpha f$ for a suitable α. ■

THEOREM 6. The norm $\|f\|$ has the following three properties:

1. $\|f\| \geqslant 0$ and $\|f\| = 0$ if and only if $f = 0$
2. $\|\alpha f\| = |\alpha| \, \|f\|$
3. $\|f + g\| \leq \|f\| + \|g\|$

The last property is known as the triangle inequality.

PROOF. Equations (1) and (2) are immediate consequences of the definition of an inner product. To prove (3) we use the Cauchy–Schwarz inequality in

$$\begin{aligned}\|f+g\|^2 &= (f+g, f+g) = \|f\|^2 + (f,g) + \overline{(f,g)} + \|g\|^2 \\ &\le \|f\|^2 + 2\,|(f,g)| + \|g\|^2 \le \|f\|^2 + 2\,\|f\|\,\|g\| + \|g\|^2 \\ &\le [\|f\| + \|g\|]^2.\end{aligned}$$

By taking square roots the result follows. One can also see that (3) reduces to an equality only if f and g are linearly dependent. ■

EXAMPLE 3. Consider the linear space X of Example 1. Let

$$(f, g) = \sum_{i=1}^{\infty} a_i \bar{b}_i.$$

It is easy to verify that the above forms an inner product. Convergence is really no problem, since only a finite number of terms in the above sum do not vanish.

EXAMPLE 4. Consider the linear space of Example 2. One can define an inner product on this space by

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx.$$

DEFINITION. Let H be an inner product space and $\{f_n\}$ a Cauchy sequence in H. Such a sequence has the property that for every $\epsilon > 0$ we can find an $N(\epsilon)$ such that

$$\|f_n - f_m\| < \epsilon \qquad \text{for} \quad n, m > N(\epsilon).$$

In other words

$$\lim_{n,m\to\infty} \|f_n - f_m\| = 0.$$

H is said to be a Hilbert space if every Cauchy sequence converges to an element in H.

It is easy to see that if such a Cauchy sequence converges it must converge to a unique element. Suppose

$$\lim_{n_k\to\infty} f_{n_k} = g$$

$$\lim_{m_k\to\infty} f_{m_k} = h$$

that is, we have two subsequences of $\{f_n\}$ such that each converges to a different element. Then

$$\begin{aligned}\|g - h\| &= \|g - f_{n_k} + f_{n_k} - f_{m_k} + f_{m_k} - h\| \\ &\le \|g - f_{n_k}\| + \|f_{n_k} - f_{m_k}\| + \|f_{m_k} - h\|.\end{aligned}$$

For n_k and m_k sufficiently large we have

$$\|g - f_{n_k}\| < \epsilon$$
$$\|f_{n_k} - f_{m_k}\| < \epsilon$$
$$\|f_{m_k} - h\| < \epsilon$$

so that

$$\|g - h\| < 3\epsilon.$$

Since g and h are independent of n_k and m_k and ϵ is arbitrary, we see that necessarily

$$\|g - h\| = 0$$

so that $g = h$.

In general an inner product space need not be a Hilbert space. For example, the space treated in Example 3 is not a Hilbert space. To see this we consider the following Cauchy sequence $\{f_n\}$ where

$$f_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right).$$

For $n > m$ we have

$$\|f_n - f_m\| = \left[\sum_{k=m+1}^{n} \frac{1}{k^2}\right]^{1/2}$$

and it follows that

$$\lim_{n,m\to\infty} \|f_n - f_m\| = 0.$$

Nevertheless, the sequence does not converge to an element in the space. Clearly, in a purely formal manner,

$$\lim_{n\to\infty} f_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots) \tag{36}$$

and none of the components of the above vanish. Since in the space in question only a finite number of components were allowed not to vanish (36) is not in the space.

We can, however, enlarge this space by adding to it all limits of Cauchy sequences. For example, we consider the space H consisting of all $f = (a_1, a_2, a_3, \ldots)$ such that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Clearly, if $f_n = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ is an element of the space X in Example 3 then

$$\lim_{n\to\infty} f_n = f.$$

Also, the set $\{f_n\}$ forms a Cauchy sequence, since

$$\|f_n - f_m\| = \left[\sum_{k=m+1}^{n} |a_k|^2\right]^{1/2} \quad \text{for} \quad n > m$$

and

$$\lim_{n,m\to\infty} \|f_n - f_m\| = 0.$$

In this fashion we see that X can be completed to a Hilbert space H.

A standard theorem in functional analysis guarantees that every inner product space X can be completed to form a Hilbert space H. Such a Hilbert space H is said to be the completion of X.

We now turn to the space $C[0, 1]$ of Example 4. Here again we can verify that we are not dealing with a Hilbert space. Consider the sequence of functions defined by

$$\begin{aligned} f_n(x) &= 1, && 0 \le x \le \tfrac{1}{2} \\ &= 1 - 2n(x - \tfrac{1}{2}), && \tfrac{1}{2} \le x \le \frac{1+n}{2n} \\ &= 0, && \frac{1+n}{2n} \le x \le 1 \\ n &= 1, 2, 3, \ldots. \end{aligned}$$

Clearly, each $f_n(x)$ is continuous and a simple calculation shows that

$$\|f_n - f_m\| \le \left[\frac{1}{n} + \frac{1}{m}\right]^{1/2}.$$

The upper estimate on the right is excessively large, but it does show that the sequence is a Cauchy sequence. Nevertheless

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= 1, && 0 \le x \le \tfrac{1}{2}. \\ &= 0, && \tfrac{1}{2} < x \le 1. \end{aligned}$$

Clearly $f_n(x)$ has a limit, but the limiting function is not in the space $C[0, 1]$. The limit function is no longer continuous.

In accordance with the general theorem this space has a completion denoted by $L_2[0, 1]$. It consists of all functions $f(x)$ for which we can find Cauchy sequences in $C[0, 1]$, $\{f_n(x)\}$ such that

$$\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|^2\, dx = 0.$$

Such functions will not be continuous in general and in fact will not even be integrable in the Riemann sense. To define their integrals over any interval in $[0, 1]$, say $[a, b]$; we can, however, proceed as follows. By definition

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$