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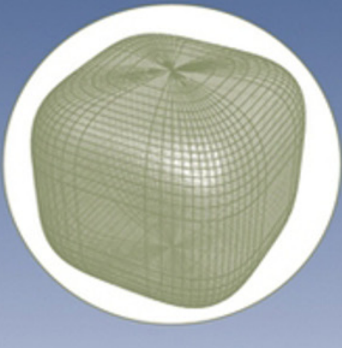
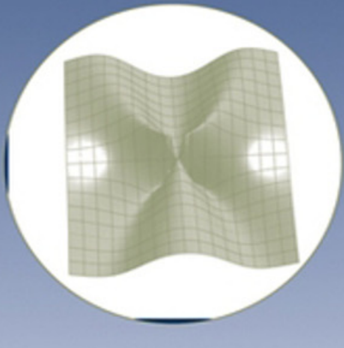
ADVANCED CALCULUS

An Introduction to Linear Analysis

Leonard F. Richardson

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An Introduction to Linear Analysis

Leonard F. Richardson

WILEY-INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada

Library of Congress Cataloging-in-Publication Data:

Richardson, Leonard F.
Advanced calculus : an introduction to linear analysis / Leonard F. Richardson.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-470-23288-0 (cloth)
1. Calculus. I. Title.
QA303.2.R53 2008
515--dc22
2008007377

Printed in Mexico

10 9 8 7 6 5 4 3 2

To Joan, Daniel, and
Joseph

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PREFACE

Why this Book was Written

The course known as Advanced Calculus (or Introductory Analysis) stands at the summit of the requirements for senior mathematics majors. An important objective of this course is to prepare the student for a critical challenge that he or she will face in the first year of graduate study: the course called Analysis I, Lebesgue Measure and Integration, or Introductory Functional Analysis.

We live in an era of rapid change on a global scale. And the author and his department have been testing ways to improve the preparation of mathematics majors for the challenges they will face. During the past quarter century the United States has emerged as the destination of choice for graduate study in mathematics. The influx of well-prepared, talented students from around the world brings considerable benefit to American graduate programs. The international students usually arrive better prepared for graduate study in mathematics—in particular better prepared in analysis—than their typical U.S. counterparts. There are many reasons for this, including (a) school systems abroad that are oriented toward teaching only the brightest students, and (b) the self-selection that is part of a student taking the step of travel abroad to study in a foreign culture.

The presence of strongly prepared international students in the classroom raises the level at which courses are taught. Thus it is appropriate at the present time, in the early years of the new millennium, for college and university mathematics departments to reconsider their advanced calculus courses with an eye toward preparing graduates for the international environment in American graduate schools. This is a challenge, but it is also an opportunity for American students and international students to learn side-by-side with, and also about, one another. It is more important than ever to teach undergraduate advanced calculus or analysis in such a way as to prepare and reorient the student for graduate study as it is today in mathematics.

Another recent change is that applied mathematics has emerged on a large scale as an important component of many mathematics departments. In applied and numerical mathematics, functional analysis at the graduate level plays a very important role.

Yet another change that is emerging is that undergraduates planning careers in the secondary teaching of mathematics are being required to major in mathematics instead of education. These students must be prepared to teach the next generation of young people for the world in which they will live. Whether or not the mathematics major is planning an academic career, he or she will benefit from better preparation in advanced calculus for careers in the emerging world.

The author has taught mathematics majors and graduate students for thirty-seven years. He has served as director of his department's graduate program for nearly two decades. All the changes described above are present today in the author's department. This book has been written in the hope of addressing the following needs.

1. Students of mathematics should acquire a sense of the unity of mathematics. Hence a course designed for senior mathematics majors should have an integrative effect. Such a course should draw upon at least two branches of mathematics to show how they may be combined with illuminating effect.

2. Students should learn the importance of rigorous proof and develop skill in coherent written exposition to counter the universal temptation to engage in wishful thinking. Students need practice composing and writing proofs of their own, and these must be checked and corrected.

3. The fundamental theorems of the introductory calculus courses need to be established rigorously, along with the traditional theorems of advanced calculus, which are required for this purpose.

4. The task of establishing the rigorous foundations of calculus should be enlivened by taking this opportunity to introduce the student to modern mathematical structures that were not presented in introductory calculus courses.

5. Students should learn the rigorous foundations of calculus in a manner that reorients thinking in the directions taken by modern analysis. The classic theorems should be couched in a manner that reflects the perspectives of modern analysis.

Features of this Text

The author has attempted to address these needs presented above in the following manner.

1. The two parts of mathematics that have been studied by nearly every mathematics major prior to the senior year are introductory calculus, including calculus of several variables, and linear algebra. Thus the author has chosen to highlight the interplay between the calculus and linear algebra, emphasizing the role of the concepts of a vector space, a linear transformation (including a linear functional), a norm, and a scalar product. For example, the customary theorem concerning uniform limits of continuous functions is interpreted as a completeness theorem for $\mathcal{C}[a, b]$ as a vector space equipped with the sup-norm. The elementary properties of the Riemann integral gain coherence expressed as a theorem establishing the integral as a bounded linear functional on a convenient function-space. Similarly, the family of absolutely convergent series is presented from the perspective that it is a complete normed vector space equipped with the l_1-norm.

2. Many exercises are offered for each section of the text. These are essential to the course. **An exercise preceded by a dagger symbol † is *cited* at some point in the text.** Such citations refer to the exercise by section and number. **An exercise preceded by a diamond symbol $\diamondsuit$ is a *hard* problem.** *If a hard problem will be cited later in the text, then there will be a footnote to say precisely where it will be cited.* This is intended to help the professor decide whether or not an exercise should be assigned to a particular class based upon his or her planned coverage for the course. Topics that can be omitted at the professor's discretion without disturbing continuity of the course are so-indicated by means of footnotes.

3. At the end of each chapter there is a brief section called *Test Yourself*, consisting of short questions to test the student's comprehension of the basic concepts and theorems. The *answers* to these short questions, and also to *other selected short questions*, appear in an appendix. There are no proofs provided among those answers to selected questions. The reason is that there are many possible correct proofs for each exercise. Only the professor or the professor's designated assistant will be able to properly evaluate and correct the student's writing in exercises requiring proofs.

4. The *Introduction* to this book is intended to introduce the student to both the importance and the challenges of writing proofs. The guidance provided in the introduction is followed by corresponding illustrative remarks that appear after the first proof in each of the five chapters of Part I of this text.

5. Whether a professor chooses to collect written assignments or to have students present proofs at the board in front of the class, each student must regularly construct and write proofs. The coherence and the presentation of the arguments must be criticized.

6. Most of the traditional theorems of elementary differential and integral calculus are developed rigorously. Since the orientation of the course is toward the role of normed vector spaces, Cauchy completeness is the most natural form of the completeness concept to use. Thus we present the system of real numbers as a Cauchy-complete Archimedean ordered field. The traditional theorems of advanced calculus are presented. These include the elements of the study of integrable and differentiable functions, extreme value theorems, Mean Value Theorems, and convergence theorems, the polynomial approximation theorem of Weierstrass, the inverse and implicit function theorems, Lebesgue's theorem for Riemann integrability, and the Jacobian theorem for change of variables.

7. Students learn in this course such concepts as those of a complete normed vector space (*real* Banach space) and a bounded linear functional. This is *not* a course in functional analysis. Rather the central theorems and examples of advanced calculus are treated as instances and motivations for the concepts of functional analysis. For example, the space of bounded sequences is shown to be the dual space of the space of absolutely summable sequences.

8. The concept of this book is that the student is guided gradually from the study of the topology of the real line to the beginning theorems and concepts of graduate analysis, expressed from a modern viewpoint. Many traditional theorems of advanced calculus list properties that amount to stating that a certain set of functions forms a vector space and that this space is complete with respect to a norm. By phrasing the traditional theorems in this light, we help the student to mentally organize the knowledge of advanced calculus in a coherent and meaningful manner while acquiring a helpful reorientation toward modern graduate-level analysis.

Course Plans that Are Supported by this Book

Part I of this book consists of five chapters covering most of the standard one-variable topics found in two-semester advanced calculus courses. These chapters are arranged in order of dependence, with the later chapters depending on the earlier ones. Though the topics are mainly the ones typically found, they have been reoriented here from the viewpoint of linear spaces, norms, completeness, and linear functionals.

Part II offers a choice of two mutually independent advanced one-variable topics: either Fourier series or Stieltjes integration. It is especially the case in Part II that each professor's individual judgment about the readiness of his or her class should guide what is taught. Some of these topics will not be for the average student, but will make excellent reading material for the student seeking honors credit or writing a senior thesis. Individual reading courses can be employed very effectively to provide advanced experience for the prospective graduate student.

In Chapter 6 the introduction of Fourier series is aided by inclusion of complex-valued functions of a real variable. This is the only chapter in which complex-valued functions appear, and with these the Hermitian inner product is introduced. The chapter includes l^2 and its self-duality, convergence in the L^2-norm,[1] the uniform convergence of Fourier series of smooth functions, and the Riemann localization theorem. The study of a vibrating string is presented to motivate the chapter.

Chapter 7, which is about Stieltjes integration, includes functions of bounded variation and the Riesz Representation Theorem, presenting the dual space of $\mathcal{C}[a, b]$ in terms of Stieltjes integration. The latter theorem of F. Riesz is the hardest one presented in this book. It is not required for the later chapters. However, it is an excellent theorem for a promising student planning subsequent doctoral study, and it requires only what has been learned previously in this course. It is a century since the discovery of the Riesz Representation Theorem. The author thinks it is time for it to take its place in an undergraduate text for the twenty-first century.

Part III is about several-variable advanced calculus, including the inverse and implicit function theorems, and the Jacobian theorems for multiple integrals. Where the first two parts place emphasis on infinite-dimensional linear spaces of functions, the third part emphasizes finite-dimensional spaces and the derivative as a linear transformation.

At Louisiana State University, Advanced Calculus is offered as a three-semester *triad* of courses.[2] The first semester is taken by all and is the starting point regardless of the subsequent choices. But the other two semesters can be taken *in either order*. This enables the Department to offer all three semesters each year, with the first semester offered in both fall and spring, and the two other courses being offered with only one of them each semester. These courses are not rushed. One must allow sufficient time for the typical undergraduate mathematics major to learn to prove theorems and to absorb the new concepts. It is the author's experience that all too often, courses in analysis are inadvertently sabotaged by packing too much subject matter into one term. It is best to teach students to take enough time to learn well and learn deeply.

A few words about testing procedures may be helpful too. At the author's institution, and at many others also, it is important to teach Advanced Calculus in a manner that is suitable for *both* those students who are preparing for graduate study in mathematics and those who are not. The author finds that it is appropriate to divide each test into two approximately equal parts: one for short questions of the type represented in the *Test Yourself* sections of this book, and the other consisting of proofs representative of those assigned and collected for homework. Although one would like each student to excel in both, there are many students who excel in one class of question but not the other. And there are indeed many students who do better in proofs than in the concept-testing short questions. Thus tests that combine both types of question provide fuller information about each student and give an opportunity for more students to show what they can do. The author always gives a choice of questions in each of the two categories: typically eight out of twelve for

[1] The L^2 norm is used here exclusively with the Riemann integral.

[2] Mathematics majors planning careers in high-school teaching take at least the first semester, while the others must take at least two of the three semesters. Those students who are contemplating graduate study in mathematics are advised strongly to take all three semesters.

the short questions, and two out of three for the proofs, for a one-hour test. The pass rate in these courses is actually high, despite the depth of the subject. Naturally, each professor will need to determine the best approach to testing for his or her own class.

It is most common for colleges and universities to offer either a single semester or else a two-semester sequence in Advanced Calculus or Undergraduate Analysis. Below the author has indicated practical syllabi for a one-semester course, as well as three alternative versions of a two-semester course. It should be understood that, depending on the readiness of the class, it may be possible to do more.

- *Single-semester course:* Sections 1.1–1.8, 2.1–2.4, 3.1–3.3, and 4.1–4.3.
- *Two-semester course leading to Stieltjes integration:*
    1. Chapters 1–3 for the first semester
    2. Chapters 4, 5, and 7 for the second semester
- *Two-semester course leading to Fourier series:*
    1. Chapters 1–3 for the first semester
    2. Chapters 4–6 for the second semester
- *Two-semester course leading to the inverse and implicit function theorems:*
    1. Sections 1.1–1.8, 2.1–2.4, 3.1–3.3, and 4.1–4.3 for the first semester
    2. Sections 8.1–8.3, 9.1–9.3, and 10.1–10.3 for the second semester
- *Three-semester course, with parts 2 and 3 interchangeable in order:*
    1. Chapters 1–3 for the first semester
    2. Either
        (a) Chapters 4–6 for the second semester or
        (b) Chapters 4, 5, and 7 for the second semester
    3. Sections 8.1–8.3, 9.1–9.3, and 10.1–10.3 for the third semester, and with Chapter 11 if there is sufficient time.

No doubt there are other possible combinations. Whatever is the choice made, the author hopes that the whole academic community of mathematicians will devote an increased number of courses to the teaching of analysis to undergraduate mathematics majors.

LEONARD F. RICHARDSON

Baton Rouge, Louisiana
August, 2007

ACKNOWLEDGMENTS

It is a pleasure to thank several colleagues at Louisiana State University who have contributed useful ideas, corrections, and suggestions. They are Professors Jacek Cygan, Mark Davidson, Charles Delzell, Raymond Fabec, Jerome Hoffman, Richard Litherland, Gestur Olafsson, Ambar Sengupta, Lawrence Smolinsky, and Peter Wolenski. Several of these colleagues taught classes using the manuscript that became this book. It is a pleasure also to thank Professor Kenneth Ross, of the University of Oregon, who provided many helpful corrections to the first printing. Of course the errors that remain are entirely my own responsibility, and further corrections and suggestions from the reader will be much appreciated.

In the academic year 1962–1963 I was a student in an advanced calculus course taught by Professor Frank J. Hahn at Yale University. His inclusion in that course of the Riesz Representation Theorem and its proof was a highlight of my undergraduate education. Though I didn't realize it at the time, that course likely was the source of the idea for this book.

Professor Hahn was a young member of the Yale faculty when I was a student in his advanced calculus course that included the Riesz theorem. He was an extraordinary and generous teacher. I became his PhD student, but his death intervened about a year later. Then Professor George D. Mostow adopted me as his student. Professor Mostow took an interest in improving undergraduate education in mathematics, having co-authored a book [14] that had as one of its goals the earlier inclusion and integration of abstract algebra into the undergraduate curriculum. I have been very fortunate with regard to my teachers. They taught lessons that grow over time like branches, integral parts of one tree. I am grateful for the opportunity to record my gratitude and indebtedness to them.

My book is intended to facilitate the integration of linear spaces, functionals and transformations, both finite- and infinite-dimensional, into Advanced Calculus. It is not a new idea that mathematics should be taught to undergraduate students in a manner that demonstrates the overarching coherence of the subject. As mathematics grows, in both pure and applied directions, the need to emphasize its unity remains a pressing objective.

Questions and observations from students over the years have resulted in numerous exercises and explanatory remarks. It has been a privilege to share some of my favorite mathematics with students, and I hope the experience has been a good one for them.

I am grateful to John Wiley & Sons for the opportunity to offer this book, as well as the course it represents and advocates, to a wider audience. I appreciate especially the role of Ms. Susanne Steitz-Filler, the Mathematics and Statistics Editor of John Wiley & Sons, in making this opportunity available. She and her colleagues provided valued advice, support, and technical assistance, all of which were needed to transform a professor's course notes into a book.

L. F. R.

INTRODUCTION

Why Advanced Calculus is Important

What is the meaning of knowledge? And what is the meaning of learning? The author believes these are questions that must be addressed in order to grasp the purpose of advanced calculus. In primary and secondary education, and also in some introductory college courses, we are asked to accept many statements or claims and to remember them, perhaps to apply them. Individuals vary greatly in temperament and are more willing or less willing to acquiesce in the acceptance of what is taught. But whether or not we are inclined to do so, we must ask responsible questions about the basis upon which knowledge rests.

Here are a few examples.

- Have we been taught accurate renditions of the history of our civilization? Is there nothing to indicate that history is presented sometimes in a biased or misleading way?

- Were we taught correct claims about the nature of the physical or biological world? Are there not examples of famous claims regarding the natural sciences, endorsed ardently, yet proven in time to be false?

- How do we know what is or is not true about mathematics? Is there no record of error or disagreement? Is there an infallible expert who can be trusted to tell correctly the answers to all questions?

- *If* there are authorities who can be trusted without doubt to instruct us correctly, what will be our fate when these authorities, perhaps older than ourselves, die? Can we not learn for ourselves to determine the difference between truth and falsehood, between valid reason and error?

In the serious study of history, one must learn how to search for records or evidence and how to appraise its reliability. In the natural sciences, one must learn to construct sound experiments or to conduct accurate observations so as to distinguish between truth and wishful thinking. And in the study of mathematics it is through logical proof by deductive reasoning that we can check our thinking or our guesswork. Learning how to confirm the foundations of our knowledge transforms us from receptacles for the claims made by others into stewards for the knowledge mankind has acquired through millennia of exertion. It is both our right as human beings and our responsibility to assume this role.

Throughout our lives, we find ourselves with the need to resolve the conflict between opposing forces. On the one hand, the human mind is impulsive, eager to leap from one spot to another that may have a clearer view. This spark is an engine of creativity. We would not be human in its absence. It is also our Achilles' heel. Training and self-discipline are required that we may distinguish the worthwhile leaps of imagination from the faulty ones.

A vital aspect of the self-discipline that must be learned by each student of mathematics is that *proofs must be written down, scrutinized step-by-step, and rewritten wherever there is doubt.* In a proof the reasoning must be solid and secure from start to finish. There is no one among us who can reliably devise a proof mentally, leaving it unwritten and unscrutinized. Indeed, mankind's capacity for wishful thinking is boundless. Discipline in the standard of logical proof is severe, and it is essential to our task.

Mathematics is not a spectator sport. It can be learned only by doing. It is necessary but never sufficient to watch proofs being constructed by an experienced practitioner. The latter activity (which includes attendance in class and active participation, as well as careful study of the text) can help one to learn good technique. But only the effort of writing our own proofs can teach each of us by trial and error how to do it. See this as not only a warning but also good news that strenuous effort in this work is effective. From more than three decades of teaching as well as personal experience, the author can assure each student that this is so. It is possible also to assure the student that through vigorous effort in mathematics the student may come to enjoy this subject very much and to relish the light that it can shed. Even a seemingly small question can be a portal to a whole world of unforeseen surprise and wonder. In this spirit it is a pleasure to welcome the student and the reader to advanced calculus.

Learning to Write Proofs: A Guide for the Perplexed Student

I want to do my proof-writing homework, but I don't know how to begin! It is an oft-heard lament. In elementary mathematics courses, the student is provided customarily with a set of instructions, or algorithms, that will lead upon implementation to the solution of certain types of problems. Thus many conscientious students have requested instructions for writing proofs. All sets of instructions for writing proofs, however, suffer from one defect: They do not work. Yet one can learn to write proofs, and there are many living mathematicians and successful mathematics students whose existence proves this point. The author believes that learning to write proofs is not a matter of following theorem-proving instructions. The answer lies rather in learning *how to study* advanced calculus. The student, having been in school for much of his or her life, may bridle at the suggestion that he or she has not learned how to study. Yet in the case of studying theoretical mathematics, that is very likely to be true. Every single theorem and every single proof that is presented in this book, or by the student's professor in class, is a vivid example of theorem-proving technique. But to benefit from these fine examples, the student must learn how to study. Mathematicians find that *the best way to read mathematics is with paper and pencil!* This means that it is the reader's task to figure out how to think about the theorem and its proof and to *write it down coherently.*

In reading the proofs of theorems in this text, or in the study of proofs presented by one's teacher in class, the student must understand that what is written is much more than a body of facts to be remembered and reproduced upon demand. Each proof has a story that guided the author in its writing. There is a beginning (the hypotheses), a challenge (the objective to be achieved), and a plan that might, with hard work, skill, and good fortune, lead to the desired conclusion. It will take time and a concerted effort for the student to learn to think about the statements and proofs of the presented theorems in this light. Such practice will cultivate the ability to read the exercises as well in a fruitful manner. With experience at recognizing the story of the proof or problem at hand, the student will be in a position to develop technique through the work done in the exercises.

The first step, before attempting to read a proof, is to read the statement of the theorem carefully, trying to get an overall picture of its content. The student should make sure he or she knows precisely the definition of each term used in the statement of the theorem. Without that information, it is impossible to understand even the claim of the theorem, let alone its proof. If a term or a symbol in the statement of a theorem or exercise is not recognized, look in the index! Write on paper what you find.

After clarifying explicitly the meaning of each term used, if the student does not see what the theorem is attempting to achieve, it is often helpful to write down a few examples to see what difficulties might arise, leading to the need for the theorem. *Working with examples is the mathematical equivalent of laboratory work for a natural scientist.* At this point the student will have read the statement of the theorem at least twice, and probably more often than that, accumulating written notes on a scratch pad along the way. Read the theorem again! Remember that in constructing a building or a bridge, it is not a waste of time to dwell upon the foundation. The author has assured many students, from freshman to doctoral level, that the way to make faster progress is to slow down—especially at the outset. If you were planning a grand two-week backpacking trip in a national park, would you simply run out of the house? Of course not—you would plan and make preparations for the coming adventure.

At this point we suppose the reader understands the statement of the theorem and wishes next to learn why the claimed conclusion is true. How does the author or teacher in class overcome the obstacles at hand? Read the whole proof a first time, *taking written notes* as to what combination of steps the author has chosen to proceed from the hypotheses to the conclusions. This first reading of the proof itself can be likened to one's first look at a road map drawn for a cross-country trip. It will give one an overall sense of the journey ahead. But taking the trip, or walking the walk, is another matter. Having noted that the journey ahead can be divided into segments, much like a trip with several overnight stops, the student should begin in earnest at the beginning. For each leg of the journey, it is important to understand thoroughly, and to write on paper, the logical justification of each individual step. There must be no magical disappearance from point A and reappearance at point B! No external authority can be substituted for the student's own understanding of each step taken. It is both the right and the responsibility of the student to understand in full detail.[3]

By studying the theorems in this book in the manner explained above, the student will cultivate the modes of thinking that will enable him or her to write the proofs that are required in the exercises.

The exercises are a vital part of this course, and the proof exercises are the most important of all. There is an answer section for *selected short-answer* exercises among the appendices of this book. It includes all the answers to the Test Yourself self-tests at the ends of the chapters. But the student will not find solutions to the proof exercises there. That is because it is not satisfactory merely to copy a written proof. Many correct proofs are possible. Only an experienced teacher can judge the correctness and the quality of the proofs you write. The student can and must depend upon his or her professor or the professor's designated assistant to read and correct proofs written as exercises.

One of the ways that a teacher can help a student is by explaining that he or she has been where the student stands. The student is not alone and can meet the challenges ahead much as his or her teacher has done before. When the author was young, he had long walks to and from school: about twenty minutes each way at a brisk pace. It was a favorite pastime during these walks to review mentally the logical structure of advanced calculus–reconstructing the proofs of theorems about Riemann integrals or uniform convergence from the axioms of the real number system. Many colleagues within mathematics, and some from theoretical physics, have shared with the author similar experiences from their own lives. It is the active engagement with a subject

[3]The student should reread this introduction before reading Remark 1.1.1, which appears after the proof of the first theorem in this book. Corresponding remarks appear following the first proof in each of the five chapters of Part I of this book.

that builds firm understanding and that incorporates the knowledge gained into ones own mind.

Experiences in life can be enjoyed only once for the first time. The student is about to embark on a mathematical adventure with advanced calculus for his or her first time. Neither the author nor your teacher can do this again. But we can wish you a wonderful journey, and we do.

PART I

ADVANCED CALCULUS IN ONE VARIABLE

CHAPTER 1

REAL NUMBERS AND LIMITS OF SEQUENCES

1.1 THE REAL NUMBER SYSTEM

During the 19th century, as applications of the differential and integral calculus in the physical sciences grew in importance and complexity, it became apparent that intuitive use of the concept of limit was inadequate. Intuitive arguments could lead to seemingly correct or incorrect conclusions in important examples. Much effort and creativity went into placing the calculus on a rigorous foundation so that such problems could be resolved. In order to see how this process unfolded, it is helpful to look far back into the history of mathematics.

Approximately 2000 years ago, Greek mathematicians placed Euclidean geometry on the foundations of deductive logic. Axioms were chosen as assumptions, and the major theorems of geometry were proven, using fairly rigorous logic, in an orderly progression. These ancient mathematicians also had concepts of numbers. They used *natural numbers*, known also as *counting numbers*, the set of which is denoted by

$$\mathbb{N} = \{1, 2, 3, \ldots, n, n+1, \ldots\}.$$

This is the endless sequence of numbers beginning with 1 and proceeding without end by adding 1 at each step. Also used were *positive rational numbers*, which we denote as

$$\mathbb{Q}^+ = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{N} \right\}.$$

These numbers were regarded as representing proportions of positive whole numbers.

Members of the Pythagorean school of geometry discovered that there was no ratio of positive whole numbers that could serve as a square root for 2. (See Exercise 1.11.) This was disturbing to them because it meant that the side and the diagonal of a square must be *incommensurable*. That is, the side and the diagonal of a square cannot both be measured as a whole number multiple of some other line segment, or *unit*. So great was these geometers' consternation over the failure of the set of rational numbers to provide the proportion between the side and the diagonal of a square that confidence in the logical capacity of algebra was diminished. Mathematical reasoning was phrased, to the extent possible, in terms of geometry.

For example, today we would express the area of a circle algebraically as $A = \pi r^2$. We could express this common formula alternatively as $A = \frac{\pi}{4} d^2$, where d is the diameter of the circle. But the ancient Greeks put it this way: The areas of two circles are in the same proportion as the areas of the *squares on their diameters*. The squares were constructed, each with a side coinciding with the diameter of the corresponding circle, and the areas of the squares were in the same proportion as the areas of the circles. Much later, in the 17th century, Isaac Newton continued to be influenced by this perspective. In his celebrated work on the calculus, *Principia Mathematica*, we can see repeatedly that where we would use an algebraic calculation, he used a geometrical argument, even if greater effort is required. The reader interested in the history of mathematics may enjoy the book *The Exact Sciences in Antiquity* by Otto Neugebauer [15] and the one by Carl Boyer [3], *The History of the Calculus*.

It took until the 19th century for mathematicians to liberate themselves from their misgivings regarding algebra. It came to be understood that the *real numbers*, the numbers that correspond to the points on an endless geometrical line, could be placed on a systematic logical foundation just as had been done for geometry nearly two thousand years earlier. Most of the axioms that were needed to prove the properties of the real number system were already quite familiar from the arithmetic of the rational numbers. There was one crucial new axiom needed: the *Completeness Axiom of the Real Number System*. Once this axiom had been added, the theorems of the calculus could be proven rigorously, and future development of the subject of *Mathematical Analysis* in the 20th century was facilitated.

Although we will not attempt the laborious task of rigorously proving every familiar property of the real number system, we will sketch the axioms that summarize familiar properties, and we will explain carefully the completeness axiom. With the latter axiom in hand, we will develop the theory of the calculus with great care. Students interested in studying the full and formal development of the real number system are referred to J. M. H. Olmsted's book [16], or to a stylistically distinctive classic by E. Landau [12].

In addition to the set $\mathbb{N}$ of natural numbers, we will consider the set $\mathbb{Z}$ of *integers,* or whole numbers. Thus

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} = \{\pm n \mid n \in \mathbb{N}\} \cup \{0\}.$$

We need also the full set of rational numbers:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

We list in Table 1.1 the axioms for a general *Archimedean Ordered Field* $\mathbb{F}$. You will observe that the set $\mathbb{Q}$ is an Archimedean ordered field. However, the set $\mathbb{R}$ of *real numbers*, which we will define in Section 1.3, will obey all the axioms for an Archimedean ordered field together with one more axiom, called the *Completeness Axiom*, which is *not* satisfied by $\mathbb{Q}$.

Table 1.1 Archimedean Ordered Field

An **Archimedean Ordered Field** $\mathbb{F}$ is a set with two operations, called addition and multiplication. There is also an *order relation*, denoted by $a < b$. These satisfy the following properties:

1. *Closure:* If a and b are elements of $\mathbb{F}$, then $a + b \in \mathbb{F}$ and $ab \in \mathbb{F}$.
2. *Commutativity:* If a and b are elements of $\mathbb{F}$, then $a + b = b + a$ and $ab = ba$.
3. *Associativity:* If a, b, and c are elements of $\mathbb{F}$, then $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$.
4. *Distributivity:* If a, b, and c are elements of $\mathbb{F}$, then $a(b + c) = ab + ac$.
5. *Identity:* There exist elements 0 and 1 in $\mathbb{F}$ such $0 + a = a$ and $1a = a$, for all $a \in \mathbb{F}$. Moreover, $0 \neq 1$.
6. *Inverses:* If $a \in \mathbb{F}$, then there exists $-a \in \mathbb{F}$ such that $-a + a = 0$. Also, for all $a \neq 0$, then there exists $a^{-1} = \frac{1}{a} \in \mathbb{F}$ such that $a\frac{1}{a} = 1$.
7. *Transitivity:* If $a < b$ and $b < c$, then $a < c$.
8. *Preservation of Order:* if $a < b$ and if $c \in \mathbb{F}$, then $a + c < b + c$. Moreover, if $c > 0$, then $ac < bc$.
9. *Trichotomy:* For all a and b in $\mathbb{F}$, exactly one of the following three statements will be true: $a < b$, or $a = b$, or $a > b$ (which means $b < a$).
10. *Archimedean Property:* If $\epsilon > 0$ and if $M > 0$, then there exists $n \in \mathbb{N}$ such that $n\epsilon > M$. (In this general context, $\mathbb{N}$ is defined as the smallest subset of $\mathbb{F}$ that contains 1 and is closed under addition.)

There is an old adage that loosely paraphrases the Archimedean Property found in the table: If you save a penny a day, eventually you will become a millionaire (or a billionaire, etc.).

From the axioms for an Archimedean ordered field, many familiar properties of the real numbers can be deduced. In particular, the behavior of all the operations used in solving equations and inequalities follows directly, with the exception that we have not established yet that roots of positive numbers, such as square roots, exist. Here we will concentrate on those properties that received less emphasis in elementary mathematics courses.

The order axioms are particularly useful for analysis. In this connection, it is important to make the following definition.

Definition 1.1.1 *We define*

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

We think of $|a|$ as representing the *distance* of a from 0 on the number line. Note that $|a|$ is always nonnegative. The absolute value satisfies a vital inequality known as the *Triangle Inequality.*

Theorem 1.1.1 *For all a and b in $\mathbb{R}$, $|a+b| \leq |a| + |b|$.*

Proof: Observe that

$$-|a| \leq a \leq |a|,$$

and

$$-|b| \leq b \leq |b|,$$

so that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|. \tag{1.1}$$

Thus, if $a + b \geq 0$,

$$|a+b| = a + b \leq |a| + |b|.$$

But if $a + b < 0$, then from the first inequality in Equation (1.1), we obtain

$$|a+b| = -(a+b) \leq |a| + |b|.$$

We see that whether $a + b$ is negative or nonnegative, we have in either case that $|a+b| \leq |a| + |b|$. ∎

Remark 1.1.1 If the student has not yet *read the Introduction*, including the discussion of *Learning to Write Proofs* on page xxiii, this should be done now. It was explained that in order to learn to write proofs, the student must learn first how to study the theorems and proofs that are presented in this book. Let us note how the remarks made there apply to the short proof of the first theorem in this book.

First we read carefully the statement of Theorem 1.1.1. We note that this is a theorem about absolute values, so we reread Definition 1.1.1 to insure that we know the meaning of this concept. Since the absolute value of a number a depends upon the sign of a, we should test the claimed inequality in the theorem with several pairs of numbers: two positive numbers, two negative numbers, and two numbers of opposite sign. The reader should *do this*, with examples of his or her choice of numbers, noting that the triangle inequality in real application gives either *equality*, if the two numbers have the same sign, or else strict inequality, if the two numbers have opposite sign. This gives us an intuitive appreciation that the triangle inequality ought to be true. Now how do we prove it? Testing more examples will not suffice, because infinitely many pairs are possible. Many correct proofs can be given, but we will discuss the one chosen by the author.

The next step in writing a proof requires some playfulness or inquisitiveness on the part of the student. In theoretical mathematics we are discouraged from following rote procedures in the hope of finding an answer without thought. To bypass thought would be to bypass mathematics itself. The student should not even consider such a route, just as he or she should not substitute a pill for a good meal.

We see by playing with the definition of absolute value that $|a|$ must be *equal* to either a or $-a$. This reminds us of what we observed when checking pairs of specific numbers of the same or opposite sign, as explained above. The playfulness appears when we choose to write this as $-|a| \leq a \leq |a|$ for all a, even though the truth of this double inequality hinges upon a being equal to either the left side or the right side. Then we do the same for b, recognizing that a and b do play symmetrical roles in the statement of the theorem. Then we add the two double inequalities, obtaining Equation (1.1). The remainder of the proof unfolds from considering that the value of $|a + b|$ hinges upon the sign of $a + b$.

This analysis of the proof of the triangle inequality is representative of what the student should do with each proof in this book, and with each proof presented in class by his or her professor. Take a fresh sheet of paper and write out a full analysis of the proof, including the perceived rationale for the course that it takes. Work on this until you are sure you understand correctly. If in doubt, ask your teacher! This is the way to learn advanced mathematics, and it is what the student must do to learn to prove theorems.

EXERCISES

1.1 Let $\epsilon > 0$. Determine how large $n \in \mathbb{N}$ must be to ensure that the given inequality is satisfied, and use the Archimedean Property to establish that such n exist.

- **a)** $\frac{1}{n} < \epsilon$?
- **b)** $\frac{1}{n^2} < \epsilon$?
- **c)** $\frac{1}{\sqrt{n}} < \epsilon$? (Assume that $\sqrt{n}$ exists in $\mathbb{R}$.)

1.2 Prove the uniqueness of the additive inverse $-a$ of a. (Hint: Suppose that

$$x + a = 0 = y + a$$

and prove that $x = y$.)

1.3 Use the Axiom of Distributivity to prove that $a0 = 0$ for all $a \in \mathbb{R}$, and use this to prove that $(-1)(-1) = 1$.

1.4 Prove that $(-1)a = -a$ for all $a \in \mathbb{R}$.

1.5 Prove the uniqueness of the multiplicative inverse a^{-1} of a for all $a \neq 0$ in $\mathbb{R}$.

1.6 Prove: For all a and b in $\mathbb{R}$, $|ab| = |a||b|$. (Hint: Consider the three cases a and b both nonnegative, a and b both negative, and a and b of opposite sign.)

1.7 Prove: For all a, b, c in $\mathbb{R}$,

$$|a - c| \leq |a - b| + |b - c|.$$

(Hint: Use the triangle inequality.)

1.8 Let $\epsilon > 0$. Find a number $\delta > 0$ small enough so that $|a-b| < \delta$ and $|c-b| < \delta$ implies $|a - c| < \epsilon$.

1.9 † Prove: For all a and b in $\mathbb{R}$,

$$\big||a| - |b|\big| \leq |a - b|.$$

Intuitively, this says that $|a|$ and $|b|$ cannot be farther apart than a and b are. (Hint: Write $|a| = |(a - b) + b|$ and use the triangle inequality. Then do the same thing for $|b|$.)

1.10 Prove or give a counterexample:

a) If $a < b$ and $c < d$, then $a - c < b - d$.
b) If $a < b$ and $c < d$, then $a + c < b + d$.

1.11 † This exercise leads in three parts to a proof that there is no rational number the square of which is 2. The reader will need to know from another source that each rational number can be written in the form $\frac{m}{n}$ in *lowest terms*. This means that m and n have no common factors other than ± 1.

a) If $m \in \mathbb{Z}$ is odd, prove that m^2 is odd.
b) If $m \in \mathbb{Z}$ is such that m^2 is even, prove that m is even.
c) Suppose there exists $\frac{m}{n} \in \mathbb{Q}$, expressed in *lowest terms*, such that

$$\left(\frac{m}{n}\right)^2 = 2.$$

Prove that m and n are both even, resulting in a contradiction.

(Hint: For this problem, if the student has not taken any class in number theory, the following definitions may be helpful. A number n is called *even* if and only if it can be written as $n = 2k$ for some integer k. A number n is called *odd* if and only if it can be written as $n = 2k - 1$ for some integer k.)

1.2 LIMITS OF SEQUENCES & CAUCHY SEQUENCES

By a *sequence* x_n of elements of a set S we mean that to each natural number $n \in \mathbb{N}$ there is assigned an element $x_n \in S$. Unless otherwise stated, we will deal with sequences of real numbers. We can think of a sequence as an endless list of real numbers, or we could equivalently think of a sequence as being a *function* whose domain is $\mathbb{N}$ and whose range lies in $\mathbb{R}$. It is very important to define the concept of the *limit* of a sequence. Intuitively, we say that x_n *approaches* the *real* number L as n *approaches infinity*, written $x_n \to L \in \mathbb{R}$ as $n \to \infty$, provided we can force $|x_n - L|$ to become as small as we like just by making n sufficiently big. This is also written with the symbols $\lim_{n\to\infty} x_n = L$. The advantage of writing the definition symbolically as follows is that this definition provides inequalities that can be solved to determine whether or not $x_n \to L$.

Definition 1.2.1 *A sequence $x_n \to L \in \mathbb{R}$ as $n \to \infty$ if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$* corresponding to ϵ *such that*

$$n \geq N \Rightarrow |x_n - L| < \epsilon.$$

If there exists a number L such that $x_n \to L$, we say x_n is convergent. *Otherwise we say that x_n is* divergent.

See Exercise 1.12.

■ **EXAMPLE 1.1**

We claim that if $x_n = \frac{1}{n}$, then $x_n \to 0$.

Proof: Let $\epsilon > 0$. We need $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\left|\frac{1}{n} - 0\right| < \epsilon.$$

That is, we need to solve the inequality $\frac{1}{n} < \epsilon$. Multiplying both sides of this inequality by the positive number $\frac{n}{\epsilon}$, we see that $\frac{1}{\epsilon} < n$. That is, if we pick $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$, then

$$n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

We know that such an N exists in $\mathbb{N}$ since ϵ and 1 are both positive. Thus there exists $N \in \mathbb{N}$ such that $N1 = N > \frac{1}{\epsilon}$ by the Archimedean Principle. ■

The student should note that the value of N does indeed correspond to ϵ. If $\epsilon > 0$ is made smaller, then N must be chosen larger.

■ **EXAMPLE 1.2**

Let $|r| < 1$. We claim that $r^n \to 0$ as $n \to \infty$.

Let $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|r^n - 0| = |r|^n < \epsilon.$$

In the special case in which $r = 0$, it would suffice to take $N = 1$. So suppose $r \neq 0$. Then we need to solve

$$\left(\frac{1}{|r|}\right)^n > \frac{1}{\epsilon}.$$

Note that we do not proceed by taking nth roots of both sides of this inequality, since we have not yet established the existence of such roots for all positive real numbers. Since $|r| < 1$, $\frac{1}{|r|} = 1 + p > 1$ for some $p > 0$. Thus

$$\begin{aligned}\left(\frac{1}{|r|}\right)^n &= (1+p)^n \\ &= (1+p)(1+p)\cdots(1+p) \\ &= 1^n + np + \cdots + p^n \\ &> np.\end{aligned}$$

By transitivity of inequalities, it would suffice to find $N \in \mathbb{N}$ such that $Np > \frac{1}{\epsilon}$. Such integers N exist because of the Archimedean property. So pick $N \in \mathbb{N}$ such $Np > \frac{1}{\epsilon}$ and we find that $n \geq N$ implies $np \geq Np > \frac{1}{\epsilon}$ so that $|r^n - 0| = |r|^n < \epsilon$.

Notice that if x_n is convergent, then after some finite number N of terms, all subsequent terms are bunched very close to one another: in fact, within ϵ of some number L. This motivates the following definition and theorem.

Definition 1.2.2 *A sequence x_n is called a* Cauchy *sequence if and only if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$,* corresponding to ϵ, *such that n and $m \geq N$ implies $|x_n - x_m| < \epsilon$.*

Theorem 1.2.1 *If x_n is any convergent sequence of real numbers, then x_n is a Cauchy sequence.*

Proof: Suppose x_n is convergent: say $x_n \to L$. Let $\epsilon > 0$. Then, since $\frac{\epsilon}{2} > 0$ as well, we see there exists $N \in \mathbb{N}$, *corresponding to* ϵ, such that $n \geq N$ implies $|x_n - L| < \frac{\epsilon}{2}$. Then, if n and $m \geq N$, we have

$$\begin{aligned}|x_n - x_m| &= |(x_n - L) + (L - x_m)| \\ &\leq |x_n - L| + |L - x_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

■

Remark 1.2.1 We make some remarks here to help the student to write his or her own detailed analysis of the proof of Theorem 1.2.1, as recommended in the introduction, on page xxiii. The student should begin with the intuitive understanding that if $x_n \to L$, then x_n will be very close to L for all sufficiently big n. The point is that we want both x_n and x_m to be so close to L that x_n and x_m must be within ϵ of one another. The student should use visualization to recognize that since x_n and x_m can be on opposite sides of L, we will need both x_n and x_m to be within $\frac{\epsilon}{2}$ of L. Then the triangle inequality for real numbers assures that x_n and x_m are no more than ϵ apart. The student should write a careful analysis of every proof in this course, whether proved in the text or by the professor in class.

■ EXAMPLE 1.3

We claim the sequence $x_n = (-1)^{n+1}$ is divergent.

In fact, if x_n *were* convergent, then x_n would have to be Cauchy. But $|x_n - x_{n+1}| \equiv 2$, for all n. Thus, if $0 < \epsilon \leq 2$, it is impossible to find $N \in \mathbb{N}$ such that n and $m \geq N$ implies $|x_n - x_m| < \epsilon$.

Definition 1.2.3 *A sequence x_n is called* bounded *if and only if there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$, for all $n \in \mathbb{N}$.*

Theorem 1.2.2 *If x_n is Cauchy, then x_n must be bounded.*

Remark 1.2.2 Observe that if x_n is convergent, then it is Cauchy, so this theorem implies that every convergent sequence is bounded.

Proof: We will show that every Cauchy sequence is bounded. In fact, taking $\epsilon = 1$, we see that there exists $N \in \mathbb{N}$ such that n and $m \geq N$ implies $|x_n - x_m| < 1$. In particular, $n \geq N$ implies

$$|x_n| - |x_N| \leq \big||x_n| - |x_N|\big| \leq |x_n - x_N| < 1$$

so that $|x_n| < 1 + |x_N|$. If we let

$$M = \max\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\},$$

making M the largest element of the indicated set of N numbers, then $|x_n| \leq M$ for all $n \in \mathbb{N}$. ■

■ EXAMPLE 1.4

If $x_n = n$, then x_n is *not* convergent.

If x_n *were* convergent, then x_n would be bounded. But for all $M > 0$, there exists $n \in \mathbb{N}$, corresponding to M, such that $n > M$ by the Archimedean Property. So x_n is not bounded.

It is also convenient to define the concepts $x_n \to \infty$ and $x_n \to -\infty$. However, ∞ is not a real number, so we have not defined anything like $|x_n - \infty|$ and thus cannot prove such a difference is less than ϵ. (Compare this with the discussion on page 9.) We adopt the following definition.

Definition 1.2.4 *We write $x_n \to \infty$ if and only if for all $M > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n > M$. Similarly, we write $x_n \to -\infty$ if and only if for all $m < 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n < m$.*

EXERCISES

1.12 † Use Definition 1.2.1 to prove that the limit of a convergent sequence x_n is unique. That is, prove that if $x_n \to L$ and $x_n \to M$ then $L = M$.

1.13 Let

$$x_n = \begin{cases} 0 & \text{if } n < 100, \\ 1 & \text{if } n \geq 100. \end{cases}$$

Prove that x_n converges and find $\lim x_n$.

1.14 Let $x_n = \frac{n-1}{n}$. Prove x_n converges and find the limit.

1.15 Let $x_n = \frac{(-1)^n}{\sqrt{n}}$. Prove x_n converges and find the limit.

1.16 Let $x_n = \frac{1}{n^2}$. Prove x_n converges and find the limit.

1.17 Let $x_n = \frac{n^2-n}{n}$. Does x_n converge or diverge? Prove your claim.

1.18 Let $x_n = \frac{(-1)^n+1}{n}$. Does x_n converge or diverge? Prove your claim.

1.19 † Prove: If $s_n \leq t_n \leq u_n$ for all n and if both $s_n \to L$ and $u_n \to L$ then $t_n \to L$ as $n \to \infty$ as well. (This is sometimes called the *squeeze theorem* or the *sandwich theorem* for sequences.)

1.20 Prove or give a counterexample:

a) $x_n + y_n$ converges if and only if both x_n and y_n converge.
b) $x_n y_n$ converges if and only if both x_n and y_n converge.
c) If $x_n y_n$ converges, then $\lim x_n y_n = \lim x_n \lim y_n$.

1.21 Let $x_n = \frac{\sin n}{n}$. Prove x_n converges, and find the limit.

1.22 † Suppose $a \leq x_n \leq b$ for all n and suppose further that $x_n \to L$. Prove: $L \in [a, b]$. (Hint: If $L < a$ or if $L > b$, obtain a contradiction.)

1.23 Suppose $s_n \leq t_n \leq u_n$ for all n, $s_n \to a < b$, and $u_n \to b$. Prove or give a counterexample: $\lim_{n\to\infty} t_n \in [a, b]$.

1.24 For each of the following sequences:

i. Determine whether or not the sequence is Cauchy and explain why.

ii. Find $\lim_{n\to\infty} |x_{n+1} - x_n|$.

a) $x_n = (-1)^n n$
b) $x_n = n + \frac{1}{n}$
c) $x_n = \frac{1}{n^2}$