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Lebesgue Measure and Integration

An Introduction

Frank Burk

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Lebesgue Measure and Integration
An Introduction

Frank Burk

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For Janet

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Preface

This book is intended for individuals seeking an understanding of Lebesgue measure and integration. As a consequence, it is not an encyclopedic reference, or a compendium, of the latest developments in this area of mathematics. Only the most fundamental concepts are presented: Lebesgue measure for R, Lebesgue integration for extended real-valued functions on R. No apologies are made for this approach, after all, it is the proper foundation for any general treatment of measure and integration. In fact, no claim to originality is made for any of the mathematics in this book, but we do accept full responsibility for any mistakes or blunders in its presentation. It is old mathematics after all (standard graduate fare for the last forty or fifty years), but particularly beautiful. It deserves a wider audience. Lebesgue measure and integration, presented properly, reveals mathematical creation in its highest form. Motivation has been the dominant concern, and understanding will be the final measure.

Where to begin? As a concession to understanding the subtleties of measure, and the effort required for such, I have taken the least upper bound axiom as a starting point. (Besides, it would be difficult, if not impossible, to improve on Landau's (1960) book, *Foundations of Analysis*.) The formal prerequisites are a basic calculus course and a course emphasizing what constitutes a proof, standard methods of proof, and the like. In reality, a curiosity for things mathematical and the "need to understand such," is both necessary and sufficient.

The arrangement of topics is standard. The historical struggle to give a rigorous definition of "area" and "area under a curve," resulting in Lebesgue measure and integration, is the subject of Chapter 1. (Tribute is paid to our mathematical ancestors by understanding and studying their results.) Mastery of this material is not necessary for subsequent chapters. After all, it is an "overview," written with the benefit of hindsight. The reader may return from time to time as she understands "measurable", "Borel", "Lebesgue Dominated Convergence," and so on. Mathematical concepts (undergraduate analysis) that are useful for the understanding of measure, measurable functions, and integration, are developed in Chapter 2. Chapter 3, measure theory, is the essence of this book. Here an elementary, but rigorous, treatment of Lebesgue measure, as a natural extension of "length of an interval" and as a subject of interest in and of itself, is presented. Set measurability is via Carathéodory's Condition. Measurable functions, motivated by the necessity of "measuring" inverse images of intervals as discussed by Lebesgue [Ma], are defined and developed in Chapter 4. The last chapter, Chapter 5, begins with the Riemann integral, developed from step functions. Replacing "step" with "simple" results in the Lebesgue integral for bounded functions on sets of finite measure. Some incisive observations and we have the celebrated convergence theorems that permit the interchange of "limit" and "integral", and justifies "Lebesgue" for those with such a need. (By the way, if at any time you are confused or lack a sense of direction, I apologize; for a solution, reread the master [Ma].) Finally, appendices A-E present other topics of beauty and inspiration to mathematicians, testament to the wonderful creativity of the human mind.

This book may be used in many ways: especially as a text for an undergraduate analysis course, first-year graduate students in statistics or probability, and other applied areas; a self-study guide to elementary analysis or as a refresher for comprehensive examinations; a supplement to the traditional real analysis course taken by beginning graduate students in mathematics.

I want to thank my good friend and colleague, Gene Meyer, for his countless hours of discussions and suggestions as to topics, and what would or would not be appropriate for a book of this nature. Accolades to Debora Naber. She had the arduous task of translating my handwriting into the final manuscript. I thank my parents, Glen and Helen Burk, whose constant encouragement has been a source of strength throughout my life. Finally, I thank my wonderful wife Janet, who somehow finds the time to encourage my dreams while rearing our five beautiful children—(Eric, Angela, Michael, Brandon, and Bryan.).

Even now there is a very wavering grasp of the true position of mathematics as an element in the history of thought. I will not go so far as to say that to construct a history of thought without profound study of the mathematical ideas of successive epochs is like omitting Hamlet from the play which is named after him. That would be claiming too much. But it is certainly analogous to cutting out the part of Ophelia. This simile is singularly exact. For Ophelia is quite essential to the play, she is very charming—and a little mad. Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

—Alfred North Whitehead

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learned as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the real and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world.

—Bertrand Russell

Lebesgue Measure and Integration

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1

Historical Highlights

Some of the major discoveries in quadratures that culminated with the Lebesgue-Young integral are presented in this chapter. Our purpose is twofold:

1. We want to acknowledge our appreciation and gratitude to the thinkers of the past. It is hoped that the student will be motivated to continue these threads that distinguish civilization from barbarism.

> Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.
>
> —Roger Bacon

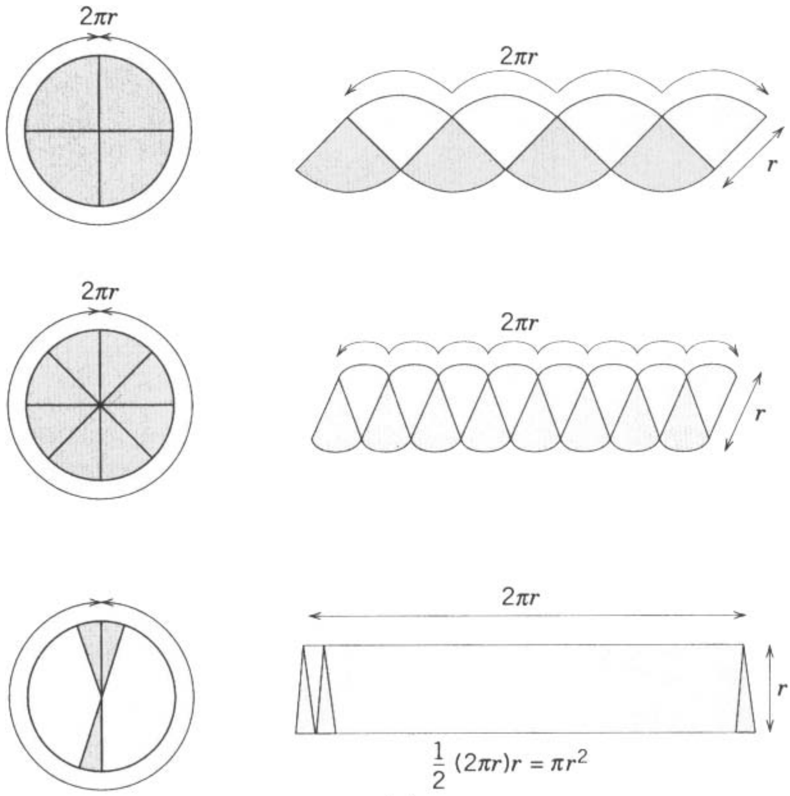
2. The student will see the process of mathematical creation and generalization as it applies to the development of the Lebesgue integral.

> Reason with a capital R = Sweet Reason, the newest and rarest thing in human life, the most delicate child of human history.
>
> —Edward Abbey

If this material is too difficult on the first reading, relax. It will make sense after Chapter 5.

1.1 REARRANGEMENTS

The figures below demonstrate the general idea of "rearranging"; in the first example, a circle rearranged into a parallelogram. This method has been known for hundreds of years.



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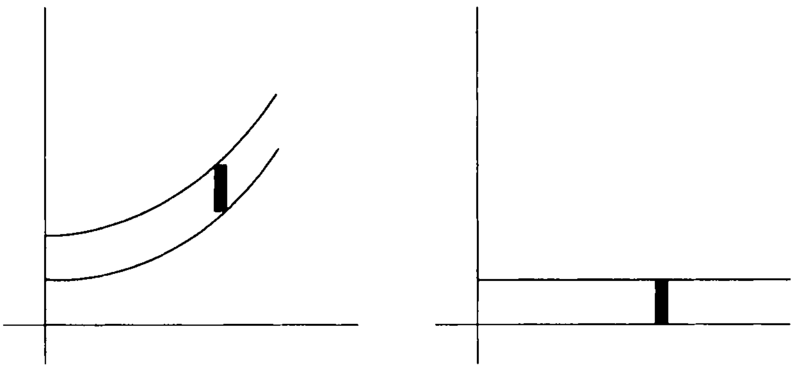

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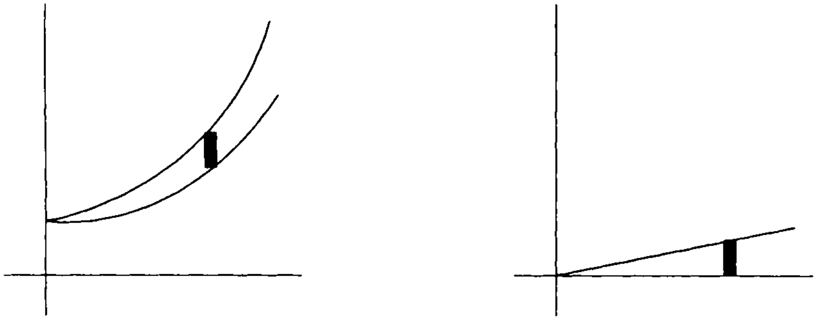

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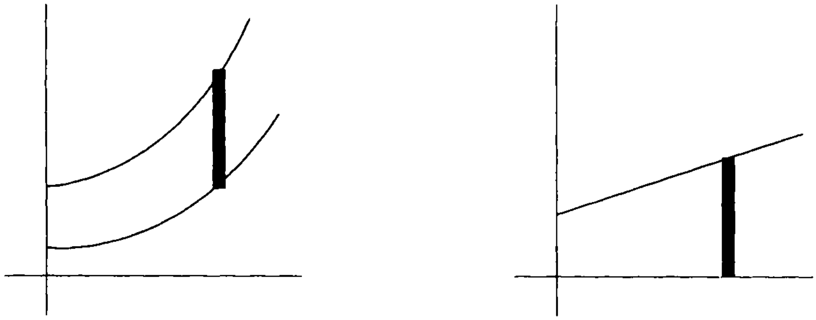

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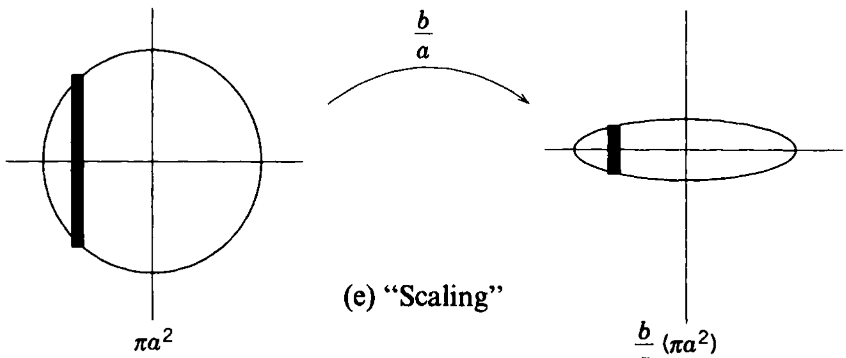


(c)

1.2 EUDOXUS (408–355 B.C.E.) AND THE METHOD OF EXHAUSTION

> “Willingly would I burn to death like Phaeton, were this the price for reaching the sun and learning its shape, its size, and its substance.”
>
> —Eudoxus

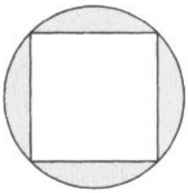
Eudoxus was responsible for the notion of approximating curved regions with polygonal regions: "truth" for polygonal regions implies "truth" for curved regions. This notion will be used to show that the areas of circles are to each other as the squares of their diameters, an obvious result for regular polygons. "Truth" was to be based on Eudoxus' Axiom:

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continuously, there will be left some magnitude which will be less than the lesser magnitude set out.

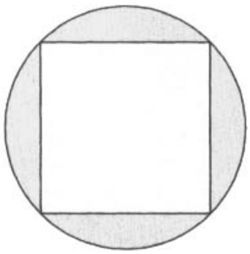
In modern terminology, let M and $\epsilon > 0$ be given with $0 < \epsilon < M$. Then form: $M, M - rM = (1 - r)M, (1 - r)M - r(1 - r)M = (1 - r)^2 M, \ldots$, where $1/2 < r \leq 1$. The axiom tells us that for n sufficiently large, say N, $(1 - r)^N M < \epsilon$, a consequence of the set of natural numbers not being bounded above.

Back to what we are trying to show: Let c, C be circles with areas a, A and diameters d, D, respectively. We want to show $a/A = d^2/D^2$, given that the result is true for polygons and given the Axiom of Eudoxus.

Assume $a/A > d^2/D^2$. Then we have $a^* < a$ so that $0 < a - a^*$ and $a^*/A = d^2/D^2$. Let $\epsilon < a - a^*$. Inscribe regular polygons of areas p_n, P_n in circles c, C and consider the areas $a - p_n$, $A - P_n$:

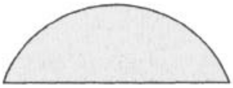


$a - p_n$

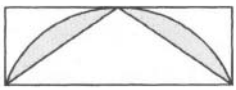


$A - P_n$

Now, double the number of sides. What is the relationship between $a - p_n$ and $a - p_{2n}$?



$a - p_n$



$a - p_{2n}$

Certainly $a - p_{2n} < 1/2(a - p_n)$. We are subtracting more than half at each stage of doubling the number of sides. From the Axiom of Eudoxus, we may determine N so that

$$0 < a - p_N < \epsilon < a - a^*, \qquad \text{that is,}$$

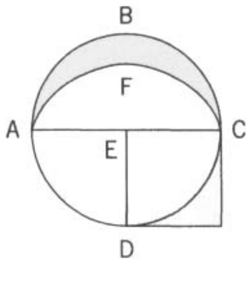
we have a regular inscribed polygon of N sides, where area $p_N > a^*$. But, $p_N/P_N = d^2/D^2$ and since $a^*/A = d^2/D^2$, we have $p_N/P_N = a^*/A$, that is, $P_N > A$. This cannot be: P_N is the area of an inscribed polygon to the circle C of area A.

A similar argument shows that a/A cannot be less than d^2/D^2:

double reductio ad absurdum.

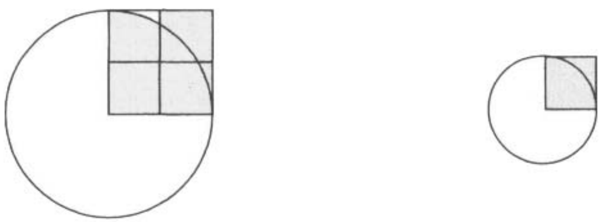
1.3 THE LUNE OF HIPPOCRATES (430 B.C.E.)

Hippocrates, a merchant of Athens, was one of the earliest individuals to find the area of a plane figure (lune) bounded by curves (circular arcs). The crescent-shaped region whose area is to be determined is shown below.



ABC and *AFC* are circular arcs with centers *E* and *D*, respectively. Hippocrates showed that the area of the shaded region bounded by the circular arcs *ABC* and *AFC* is exactly the area of the shaded square whose side is the radius of the circle. The argument depends on the following assumption:

(a) The areas of two circles are to each other as the squares of the radii:



(*a*)

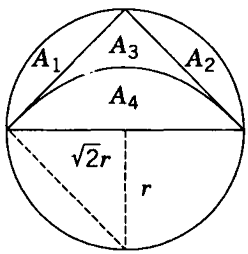
From this assumption we conclude that (b) the sectors of two circles with equal central angles are to each other as the squares of the radii:

(b)

(c) The segments of two circles with equal central angles are to each other as the squares of the radii:

(c)

Hippocrates' argument proceeds as follows:



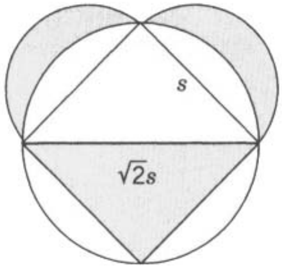
From (c), $A_1/A_4 = r^2/(\sqrt{2}r)^2 = 1/2$. Hence $A_1 = 1/2\, A_4$ and $A_2 = 1/2\, A_4$ and thus $A_1 + A_2 = A_4$.

$$\begin{aligned}\text{The area of the lune} &= A_1 + A_2 + A_3 \\ &= A_4 + A_3 \\ &= \text{area of triangle} \\ &= \tfrac{1}{2}(\sqrt{2}r)(\sqrt{2}r) \\ &= r^2 \\ &= \text{area of the square.}\end{aligned}$$

The reader may use similar reasoning on these "lunes":

1.

2.



He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.

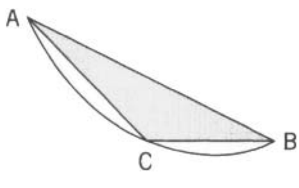
—Plato

1.4 ARCHIMEDES (287–212 B.C.E.)

It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius, . . .

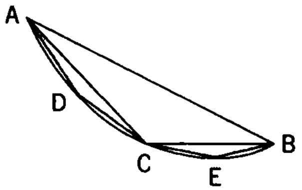
—Plutarch

This masterpiece of mathematical reasoning is due to one of the greatest intellects of all time, Archimedes of Syracuse. He shows that the area of the parabolic segment is 4/3 that of the inscribed triangle ACB. (The symbol $\triangle$ will denote "area of".)

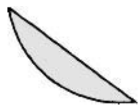


The argument proceeds as follows: the combined area of triangle ADC and BEC is one-fourth the area of triangle ACB, that is,

$$\triangle ADC + \triangle BEC = \frac{1}{4}\triangle ACB.$$



Repeating the process, trying to "exhaust" the area between the parabolic curve and the inscribed triangles, we have:

The area of the parabolic segment = 

$$= \triangle ACB + \frac{1}{4}(\triangle ACB) + \frac{1}{4}\left(\frac{1}{4}(\triangle ACB)\right) + \cdots$$

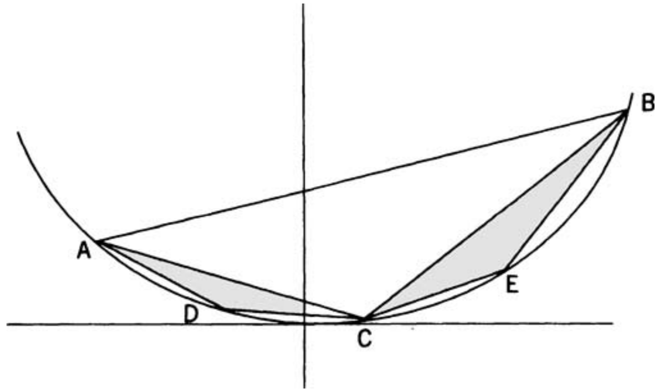
$$= \triangle ACB\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right)$$

$$= \frac{4}{3}\triangle ACB.$$

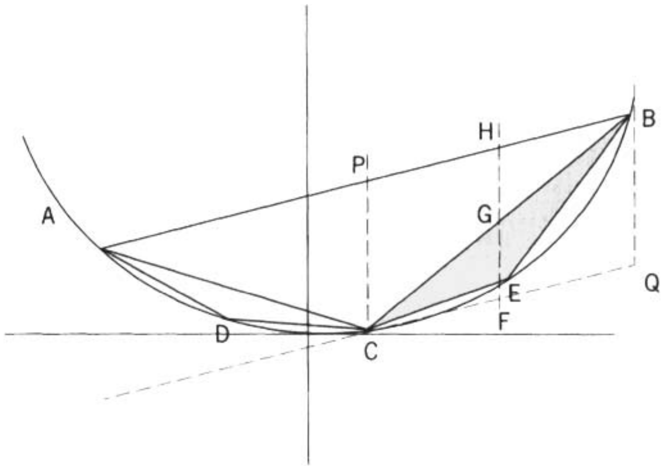
We argue that

$$\triangle ADC + \triangle BEC = \frac{1}{4}\triangle ACB$$

for the parabola $y = ax^2, \qquad a > 0.$



The reader should show the tangent line at C is parallel to AB and that the vertical line through C bisects AB at P. We need to show $\triangle BEC = 1/4\triangle BCP$. Complete the parallelogram:



We note:

1. $\triangle CEG = \triangle BEG$ (equal height and base)
2. $\triangle HGB = \frac{1}{4}\triangle BCP$.

Thus, we must show

$$\triangle CEG + \triangle BEG = \triangle HGB,$$

or that

$$\triangle BEG = \frac{1}{2}\triangle HGB.$$

This will be accomplished by showing $FE = 1/4\,FH = 1/4\,QB$. Since

$$\begin{aligned} FE &= a((X_C + X_B)/2)^2 - \left[aX_C^2 + 2aX_C \times \frac{1}{2}(X_B - X_C)\right] \\ &= \frac{1}{4}a(X_B - X_C)^2, \end{aligned}$$

and

$$\begin{aligned} QB &= aX_B^2 - \left[aX_C^2 + 2aX_C(X_B - X_C)\right] \\ &= a(X_B - X_C)^2, \end{aligned}$$

we are done.

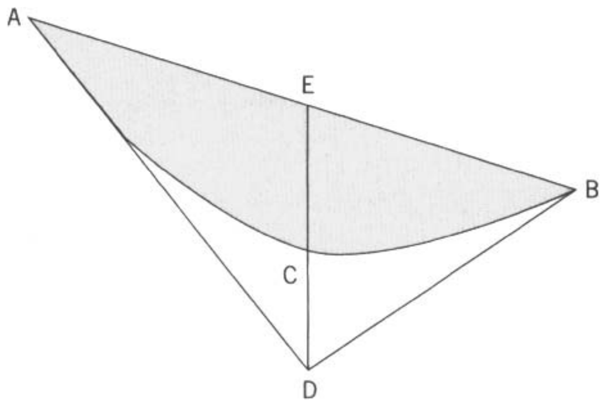
...there was far more imagination in the head of Archimedes than in that of Homer.

—Voltaire

Archimedes will be remembered when Aeschylus is forgotten because languages die and mathematical ideas do not.

—G.H. Hardy

The reader may show that the area of the parabolic segment is 2/3 the area of the circumscribed triangle ADB formed by the tangent lines to the parabola at A and B with base $AB(EC = CD)$.



1.5 PIERRE FERMAT (1601–1665): $\int_0^b x^{p/q}dx = b^{p/q+1}/(p/q+1)$

It appears that Fermat, the true inventor of the differential calculus,...

—Laplace

The Italian mathematician Cavalieri demonstrated (1630's) that

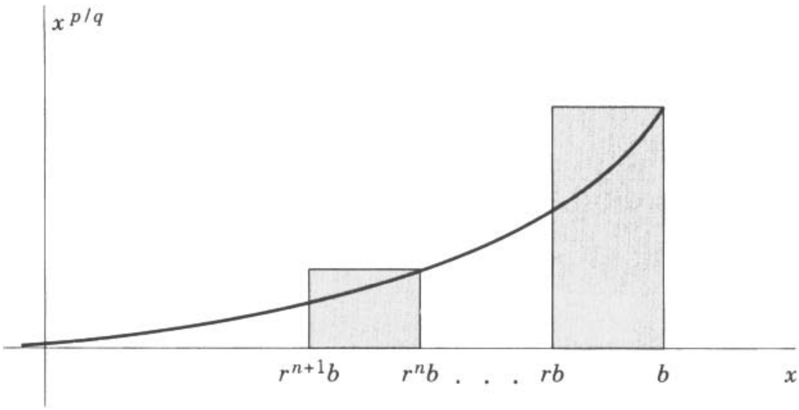
$$\int_0^b x^n\,dx = \frac{b^{n+1}}{n+1}$$

for $n = 1, 2, \ldots, 9$. But it was Fermat who was able to show

$$\int_0^b x^{\frac{p}{q}}dx = \frac{b^{p/q+1}}{\frac{p}{q}+1},$$

where p/q is a positive rational number.

Fermat divided the interval $[0, b]$ into an **infinite** sequence of subintervals with endpoints (heretofore a finite number of subintervals of equal width) br^n, $0 < r < 1$, and erected a rectangle of height $(br^n)^{p/q}$ over the subinterval $[br^{n+1}, br^n]$ (see below).



Let S_r denote the sum of the areas of the exterior rectangles. We have

$$
\begin{aligned}
S_r &= (b - br)b^{\frac{p}{q}} + (br - br^2)(br)^{\frac{p}{q}} + \cdots + (br^n - br^{n+1})(br^n)^{\frac{p}{q}} + \cdots \\
&= b^{\frac{p}{q}+1}(1 - r)\left[1 + r^{\frac{p}{q}+1} + r^{(\frac{p}{q}+1)2} + \cdots + r^{(\frac{p}{q}+1)n} + \cdots\right] \\
&= \frac{b^{\frac{p}{q}+1}(1 - r)}{(1 - r^{\frac{p}{q}+1})} \\
&= b^{\frac{p}{q}+1}\,\frac{\left[1 - (r^{\frac{1}{q}})^q\right]}{(1 - r^{\frac{1}{q}})}\;\frac{(1 - r^{\frac{1}{q}})}{\left[1 - (r^{\frac{1}{q}})^{p+q}\right]} \\
&= b^{\frac{p}{q}+1}\,\frac{\left(1 + r^{\frac{1}{q}} + \cdots + r^{\frac{q-1}{q}}\right)}{\left(1 + r^{\frac{1}{q}} + \cdots + r^{\frac{p+q-1}{q}}\right)} \\
&\longrightarrow b^{\frac{p}{q}+1}\,\frac{q}{p+q} \qquad \text{as } r \to 1 \\
&= \frac{b^{\frac{p}{q}+1}}{\frac{p}{q}+1}.
\end{aligned}
$$

. . . a master of masters.

—E.T. Bell

1.6 GOTTFRIED LEIBNITZ (1646–1716), ISSAC NEWTON (1642–1723)

> Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.
>
> —Gottfried Leibnitz

> Nature and Nature's laws lav hid in night; God said, "Let Newton be!" and all was light.
>
> —Alexander Pope

> The capital discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz.
>
> —Sophus Lie

During the seventeenth and eighteenth centuries the integral was thought of in a descriptive sense, as an antiderivative, due to the beautiful Fundamental Theorem of Calculus (FTC), as developed by Leibnitz and Newton. The ease of this method for specific functions probably induced a sense of euphoria, as generations of calculus students can attest to after struggling through Riemann sums. A particular function f on $[a, b]$ was integrated by finding an antiderivative F so that $F' = f$ or by finding a power series expansion and using the FTC to integrate termwise. The Leibnitz-Newton integral of f was $F(b) - F(a)$, that is,

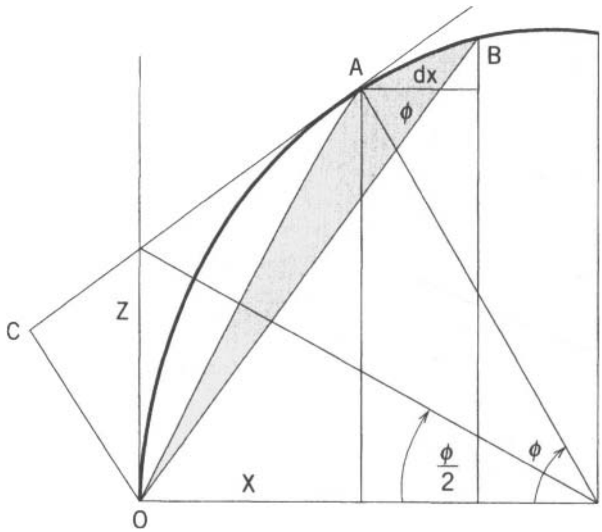
$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F' = f$.

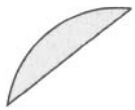
We give an argument of Leibnitz and a result of Newton to illustrate the power of these geniuses.

$$\text{Leibnitz}: \ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Take the quarter circle $(x-1)^2 + y^2 = 1,\ 0 \leq x \leq 1$, whose area is $\pi/4$:



Leibnitz determines the area of the circular sector



by dividing it into infinitesimal triangles OAB, A and B two close points on the circle, and summing. So, how to estimate the area of OAB, henceforth ΔOAB. Construct the tangent to the circle at A, with a perpendicular at C passing through the origin. Then $\Delta OAB \approx 1/2\, AB \times OC$. By similar triangles, $AB/dx = z/OC$, so $\Delta OAB = 1/2\, z\, dx$. Observe

$$x = 1 - \cos\phi = 2\sin^2\frac{\phi}{2} \qquad \text{and} \qquad z = \tan\frac{\phi}{2}, \qquad \text{that is,}$$