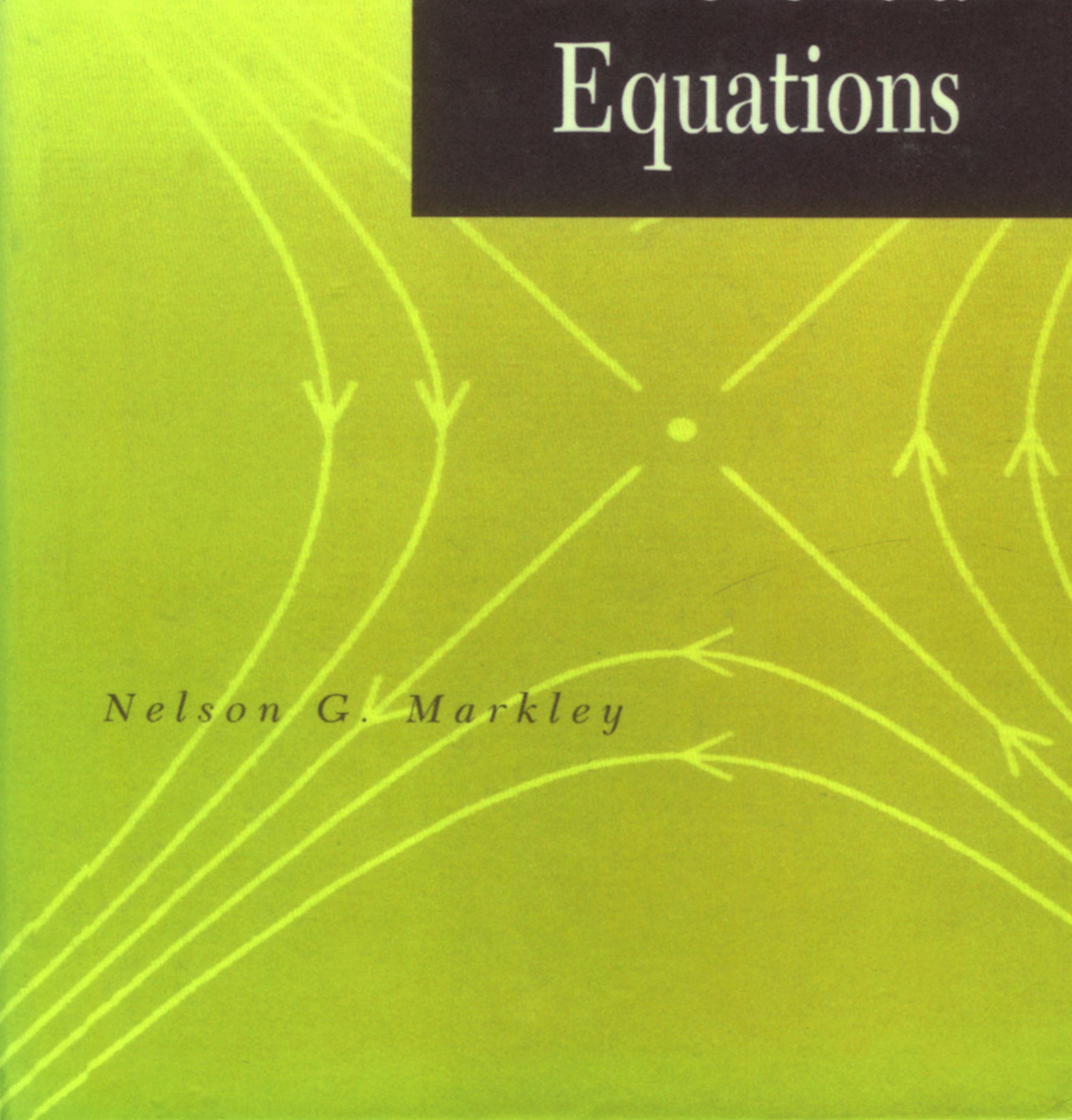
Pure and Applied Mathematics:
A Wiley-Interscience Series of Texts, Monographs, and Tracts

Principles of Differential Equations



Nelson G. Markley

This page intentionally left blank

Principles of
Differential Equations

PURE AND APPLIED MATHEMATICS

A Wiley-Interscience Series of Texts, Monographs, and Tracts

A complete list of the titles in this series appears at the end of this volume.

Principles of Differential Equations

Nelson G. Markley
Provost Emeritus
Lehigh University
Bethlehem, PA
and
Professor Emeritus
University of Maryland
College Park, MD

WILEY-INTERSCIENCE
A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Markley, Nelson Groh
Principles of differential equations / Nelson G. Markley.
p. cm — (Pure and applied mathematics: a Wiley-Interscience series of texts, monographs, and tracts)
Includes bibliographical references and index.
ISBN 0-471-64956-2 (cloth)
1. Differential equations. I. Title. II. Pure and applied mathematics (John Wiley & Sons : Unnumbered)

QA371.M27 2004
515'.35—dc22 2004040890

Printed in the United States of America.

10 9 8 7 6 5 4 3 2

Contents

Preface

Differential equations is an old but durable subject that remains alive and useful to a wide variety of engineers, scientists, and mathematicians. The purpose of this book is to provide an introductory graduate text for these consumers. It is intended for classroom use or self-study. The goal is to provide an accessible and concrete introduction to the main principles of ordinary differential equations and to present the material in a modern and rigorous way. The intent of this goal is to provide the solid foundation that will enable a reader to learn and understand other parts of the subject easily and to encourage them to learn more about differential equations and dynamical systems.

The study of differential equations began with the birth of calculus, which dates to the 1660s. Part of Newton's motivation in developing calculus was to solve problems that could be attacked with differential equations. For example, an early triumph of differential equations was Newton's demonstration that Kepler's empirical laws of planetary motion could be derived from Newton's laws of motion using differential equations. Now, with over 300 years of history, the subject of differential equations represents a huge body of knowledge including many subfields and a vast array of applications in many disciplines. It is beyond exposition as a whole. Instead, the right question to ask is what are the principles of differential equations that a serious user should know and understand today?

Principles of Differential Equations is my answer to this question. It looks at ordinary differential equations from the viewpoint of important principles. Although the word "principle" is probably overused in the academic world and may be a bit trite, it is used here seriously in the sense of "a basic or essential quality or element determining intrinsic nature or characteristic behavior." Each section presents a coherent picture of a circle of ideas that illustrates a key principle in the study of differential equations. The overarching questions driving the theory are discussed and the value and limitations of results are explained. Throughout, the book a concerted effort is made to tie the pieces together and give the reader a coherent and unified sense of the subject.

Principles of Differential Equations is also largely about the qualitative theory of ordinary differential equations. Qualitative theory refers to the study of the behavior of solutions without determining explicit formulas for the solutions. It originated with Poincaré at the beginning of the twentieth century and, in my judgment, has been the most important theme of ordinary differential equations in that century. Consequently, very little attention is paid to techniques for finding analytic formulas for solutions. The emphasis is on the general properties of the solutions of ordinary differential equations from simple existence of solutions to the remarkable behavior of Hopf bifurcations.

Another important development in the twentieth century was the study of dynamical systems. Since my research has always been in dynamical systems, this book naturally has a dynamical systems perspective. In ordinary differential equations, the dynamical systems approach amounts to a shift in emphasis from finding the solution of a particular problem to studying all the solutions of a differential equation at once, and it is closely linked to the qualitative point of view. Once the existence of a global solution containing all solutions is established at the end of Chapter 1, it plays a central role in the remaining chapters. Furthermore, various branches of modern dynamical systems have roots in ordinary differential equations and are briefly discussed with suggested introductory references at appropriate points in the text.

Since the broad plan of the book is to expose the reader to a range of important ideas and basic results, the focus is on core concepts and theorems that apply to large classes of differential equations and not on being encyclopedic on any topic. This means many things, some more important than others, have been deliberately omitted. There will be, as there should, instructors who strongly disagree with my choices of what to include and what not to include. I would simply invite them to supplement the material in this book with a series of well-prepared lectures on their favorite missing topic and begin expanding their students' horizons.

I have strived to make this volume as complete and self-contained as possible with minimal prerequisites, which are discussed at length in the next three paragraphs. Except for obvious exceptions like the Jordan curve theorem in Chapter 6, stated results are followed by rigorous proofs or left to the reader as straightforward exercises. Theorems and propositions are numbered consecutively in each chapter; lemmas and corollaries are unnumbered. Because the results build on each other, there are many cross-references to help the reader follow the arguments and see how the pieces fit together. There are also approximately 250 exercises that illustrate the material with specific differential equations, fill in gaps, or slightly extend the theory.

To make the book as accessible as possible, the prerequisites have been kept to a minimum. They are primarily undergraduate real analysis of one variable (sometimes called advanced calculus) and introductory linear algebra. For a mathematically capable student one semester of each should suffice, but would require the student to spend more time mastering Chapter 1, which is both challenging and essential. In particular, to gain an understanding of how the fundamentals fit together, some readers may find it beneficial to skip the proofs in a first reading of Chapter 1 and possibly the first two section of Chapter 2 and then go back and study the proofs. A number of advanced topics from both analysis and linear algebra that are less likely to be familiar to a reader are included with proofs when needed in text.

From analysis it is assumed that the reader understands epsilon–delta proofs and knows the standard concepts and results for both the real numbers (density of the rational numbers, Bolzano–Weiestrass theorem, convergence of sequences, Cauchy sequences, etc.) and real-valued function of one variable (limit theorems, continuity, uniform continuity, the intermediate-value theorem, the mean-value theorem, fundamental theorem of calculus, Taylor's formula with remainder, uniform convergence of a sequence of functions, etc.).

It is also assumed that the reader is comfortable working with functions of several variables, their partial derivatives, and integrals. To facilitate the shift from just working with functions of several variables to doing rigorous analysis with them, Section 1.1 and the exercises following it provide a rigorous but brief introduction to the analysis of functions of several variables. The approach to functions of several variables is topological, and depending on a readers background may require more or less time to master.

The prerequisites from linear algebra are a basic knowledge of matrix algebra for real and complex matrices, finite dimensional vector spaces, and linear transformations. From matrix algebra it is assumed that the reader is familiar with matrix calculations including the determinants and inverses of matrices and with systems of linear equations. The vector space prerequisites are subspaces, linear independence, basis, and dimension. Finally, the reader should be familiar with the relationship between matrices and linear transformations and with the nullity, rank, and eigenvalues of a linear transformation, but these concepts are also reviewed when they first occur in the text.

The bibliography consists entirely of books and is longer than would be absolutely necessary. The intent is to provide the reader with a rich list of books that are the next steps toward the frontiers of differential equations and dynamical systems. Many of them have extensive bibliographies of important current and historical research papers in a wide variety of journals. A number of them are excellent introductions to closely related fields. All are appropriately referenced at some point.

This volume grew out of my lecture notes for the introductory graduate course in differential equations at the University of Maryland. The students were typically first or second year graduate students in the mathematics or applied mathematics programs and a few graduate students from the engineering and physics programs. There was a large variation in their backgrounds and the challenge was to engage all of them in the material. This experience more than anything else shaped my thinking about what constitutes a coherent and accessible core to the modern theory of differential equations. My lecture notes over the years contained a variety of different topics that enriched the course but were eventually discarded because they were not really central to the development of the subject. The material was reorganized and the proofs rethought every time I taught the course. Preparing this manuscript was the final distillation.

Over the years I have learned about differential equations from a great variety of books at all levels. I am deeply indebted to the authors of all these books for everything I learned from them. Collectively, they shaped my perspective of the subject and provided a foundation for my lectures, notes, and eventually this book.

I want to thank all my colleagues and friends in the Department of Mathematics at the University of Maryland for affording me the opportunity to regularly teach the graduate course on differential equations in a stimulating mathematical environment during my many years in the department. Numerous discussions about the course content with colleagues in all areas and the semiannual preparation of the qualifying exam in differential equations with the dynamics group were particularly valuable to me as I developed my own approach to the subject. I also want to thank the Department of Mathematics for typing and reproducing an early version of my notes and Jay Alexander for using it as the textbook for the course and for his many insightful comments. Special thanks go to Mary Vanderschoot for reading the final manuscript and doing a wonderful job finding all kinds of little errors that needed to be corrected. Finally, I am particularly grateful to Lehigh University for a very generous sabbatical leave that allowed me to complete this book.

I have tried to write the kind of book I would have enjoyed reading and benefitted from as a graduate student. It is my hope that it will fill that role for others.

Nelson G. Markley

Chapter 1

Fundamental Theorems

The subject of this book is ordinary differential equations of the form $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ where $\mathbf{f}(t, \mathbf{x})$, is a continuous function, $\mathbf{x}$ is a vector, and $\dot{\mathbf{x}}$ denotes the ordinary derivative with respect to the single variable t. The vector $\mathbf{x}$ is often thought of as a space variable and t as time.

Differential equations is an old subject whose long history goes back to Newton and Leibnitz and is tightly interwoven with that of calculus and classical physics. During the nineteenth century, the foundations of differential equations were more rigorously established, and in the twentieth century, it has continued to grow and develop in important new directions. The goal of this book is to provide an accessible concrete introduction to ordinary differential equations that is both modern and rigorous.

The purpose of this first chapter is to prove the basic facts on which the many branches of ordinary differential equations rest. These foundational results—existence, uniqueness, continuation, numerical approximation, and continuity in initial conditions—are akin to the axioms of abstract subjects like group theory. They are always with us and their use in general or specific questions becomes automatic.

To help make the proof of these fundamental results more accessible, the first section of the chapter provides a bridge from the core theoretical ideas of calculus to the analysis of functions of several variables and some specific results needed later in this chapter. Such a bridge cannot meet every reader's needs, but from it most readers should be able to build on their past knowledge to understand better the mathematical framework for studying differential equations.

The most fundamental question is the existence of solutions of an ordinary differential equation, because without solutions there is no subject of differential equations. The question of existence of solutions is addressed by requiring only that $\mathbf{f}(t, \mathbf{x})$ be continuous, although the proofs with more restrictive hypotheses are technically simpler. The advantage of requiring only continuity is that it provides a simple well-understood general context for studying principles of differential equations.

The proof of the main result about the existence of solutions to an ordinary differential equation raises three basic questions. When are solutions to a differential equation uniquely determined? Is there a constructive method for approximating solutions? How far can solutions be extended? These questions will be addressed in subsequent sections on Uniqueness, Numerical Methods, and Continuation. A key hypothesis running through these sections and the final section is that $\mathbf{f}(t, \mathbf{x})$ satisfy a Lipschitz condition. Consequently, understanding this hypothesis and when it holds will be important.

The last section focuses on the collective behavior of all the solutions of a differential equation. The key result, continuity in initial conditions, shows that all the solutions of a differential equation are bound together in one continuous function. In other words, solutions that start sufficiently close stay close over a finite interval of time. This result along with the previous sections provide a good set of fundamental tools for studying the solutions of ordinary differential equations.

1.1 Preliminaries

Before beginning the study of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, it is necessary to set up some notation, review some basic facts, and describe the context for the study of such differential equations. In addition, a key theorem that is needed for existence will be proved at the end of the section.

The real numbers will always be denoted by $\mathbb{R}$, and $\mathbb{R}^m$ will denote m-dimensional Euclidean space, that is, $\mathbb{R}^m$ consists of all m-tuples of real numbers or

$$\mathbb{R}^m = \{(x_1, x_2, \ldots, x_m) : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, m\}.$$

Boldface type will be used consistently to denote elements of $\mathbb{R}^m$, $m > 1$, that is, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. Elements of $\mathbb{R}^m$ can be thought of as points in Euclidean space or as vectors pointing from the origin to $\mathbf{x}$. Furthermore, boldface type will also be used to denote functions whose values are in $\mathbb{R}^m$ or what are commonly called vector valued functions.

The best approach to the study of $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ is to use the topological ideas of open sets, closed sets, compact sets, and the norms used to define them. Consequently, open, closed, and compact sets of $\mathbb{R}^m$ will be used frequently, and so introducing these topological ideas is a natural starting point.

The *Euclidean norm* on $\mathbb{R}^m$ is defined by

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. The *distance* between $\mathbf{x}$ and $\mathbf{y}$ is defined to be $\|\mathbf{x}-\mathbf{y}\|$. This is the standard *Euclidean distance* between $\mathbf{x}$ and $\mathbf{y}$. In particular, $\|\mathbf{x}\|$ is just the Euclidean distance from $\mathbf{x}$ to $\mathbf{0} = (0, 0, \ldots, 0)$ and satisfies the following conditions for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$ and α in $\mathbb{R}$:

(a) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(b) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$; and

(c) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

The last condition is called the *triangle inequality.*

The proof of the triangle inequality and other useful elementary facts that may or may not be familiar to the reader are included in the exercises at the end of this section.

Remark *For* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^m$,

$$\big|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,\big| \le \|\mathbf{x}-\mathbf{y}\|.$$

Proof. By the triangle inequality $\|\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}\| \le \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\|$ or $\|\mathbf{x}\| - \|\mathbf{y}\| \le \|\mathbf{x}-\mathbf{y}\|$. Similarly, $\|\mathbf{y}\| - \|\mathbf{x}\| \le \|\mathbf{y}-\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}\|$ and the conclusion follows. □ (The symbol □ will be used to indicate the end of a proof.)

A set U in $\mathbb{R}^m$ is an *open set* if for every $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that

$$\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\| < \varepsilon\} \subset U.$$

It is easy to show that the *Euclidean ball*

$$\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\| < r\}$$

of radius r with center at $\mathbf{x}$ is itself an open set. Moreover, the union of open sets is open and the intersection of a finite number of open sets is open.

A subset F of $\mathbb{R}^m$ is a *closed set* if its complement,

$$\mathbb{R}^m \setminus F = \{\mathbf{x} : \mathbf{x} \notin F\}$$

is open. Not surprisingly, the closed Euclidean ball

$$\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\| \le r\}$$

is a closed set. It follows from the above remarks about open sets that the intersection of closed sets is closed and the finite union of closed sets is closed. Note that $\mathbb{R}^m$ and the empty set, ϕ, are sets that are both open and closed.

Using the distance function $\|\mathbf{x}-\mathbf{y}\|$ instead of the usual absolute value $|x-y|$, it is easy to define the convergence of sequences in $\mathbb{R}^m$. A sequence of points $\mathbf{x}_k$ in $\mathbb{R}^m$ converges to $\mathbf{y}$ if given $\varepsilon > 0$, there exists $N > 0$ such that $\|\mathbf{x}_k - \mathbf{y}\| < \varepsilon$ when $k \ge N$. Moreover, the standard result from advanced calculus that a sequence of real numbers converges to a real number if and only if it is a Cauchy sequence extends to $\mathbb{R}^m$. (See *Exercise 9.*)

The idea of a compact set is more subtle. A set C is a *compact set* provided that whenever $\{U_\lambda\}$, $\lambda \in \Lambda$, is a family of open sets indexed by Λ such that

$$C \subset \bigcup_{\lambda\in\Lambda} U_\lambda$$

then there exists a finite set of indices $\lambda_1, \ldots, \lambda_k$ such that

$$C \subset \bigcup_{i=1}^{k} U_{\lambda_i}.$$

The essential theorem about compact sets in $\mathbb{R}^m$ is the well-known Heine-Borel theorem.

Theorem 1.1 (Heine-Borel) *A subset C of $\mathbb{R}^m$ is compact if and only if it is both closed and bounded.*

Proof. First assume C is compact. If C is not closed, there exists a sequence $\mathbf{x}_k$ in C converging to $\mathbf{y} \notin C$. (See *Exercise 6.*) For every $r > 0$ set

$$U_r = \mathbb{R}^m \setminus \{\mathbf{x} : \|\mathbf{x} - \mathbf{y}\| \leq r\} = \{\mathbf{x} : \|\mathbf{x} - \mathbf{y}\| > r\}.$$

Then each U_r is open,

$$C \subset \bigcup_{r>0} U_r,$$

but C is not contained in a finite union of the sets U_r, $r > 0$, contradicting the assumed compactness of C. Therefore, C must be closed.

If C is not bounded, there exists a sequence $\mathbf{x}_k \in C$ such that $\|\mathbf{x}_k\|$ goes to infinity. Set $U_r = \{\mathbf{x} : \|\mathbf{x}\| < r\}$ and obtain a contradiction as above. This completes the proof of the first half of the theorem.

Now, assume C is closed and bounded and suppose

$$C \subset \bigcup_{\lambda \in \Lambda} U_\lambda,$$

where each U_λ is open. Every open set is a union of Euclidean balls of the form

$$\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < r\}$$

where r is rational and the center $\mathbf{x}$ has rational coordinates, and there are only countably many such sets. (See *Exercises 11 and 12.*) Consequently, it suffices to consider

$$C \subset \bigcup_{i=1}^{\infty} U_i,$$

where $U_i = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}_i\| < r_i\}$, r_i is rational, and $\mathbf{x}_i$ has rational coordinates and to show that

$$C \subset \bigcup_{i=1}^{N} U_i,$$

for some N.

Suppose this does not hold. Then for each integer k there exists

$$\mathbf{x}_k \in C \setminus \bigcup_{i=1}^{k} U_i.$$

Consequently, the coordinates of the sequences of $\mathbf{x}_k$, $k = 1, \ldots$, are bounded because C is bounded. A bounded sequence of real numbers has a convergent subsequence by the Bolzano-Weierstrass theorem. (We assume the reader is familiar with the Bolzano-Weierstrass theorem from advanced calculus or real analysis.) In particular, there exists a subsequence with convergent first coordinates. It in turn has a subsequence with both first and second coordinates converging. Do this m-times to get a subsequence $\mathbf{x}_{k_j}$ converging to $\mathbf{y}$. Because C is closed, $\mathbf{y} \in C$. Therefore, $\mathbf{y} \in U_k$ for some k and $\mathbf{x}_{k_j} \in U_k$ for large j. This contradiction completes the proof. □

Consider a function $\mathbf{f} : W \to \mathbb{R}^n$, where W is an open subset of $\mathbb{R}^m$ and m and n are arbitrary positive integers. The function $\mathbf{f}$ is *continuous* at $\mathbf{x} \in W$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| < \varepsilon$ whenever $\|\mathbf{y} - \mathbf{x}\| < \delta$. Although continuity is a point property, we will only be using functions that are continuous at every point of their domain. In this context, it is easy to prove the following: $\mathbf{f}$ is continuous at every point of W if and only if $\mathbf{f}^{-1}(U) = \{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in U\}$ is open for every open set U of $\mathbb{R}^n$. The next theorem links compactness and continuity by showing that the continuous image of a compact set is compact.

Proposition 1.2 *Let $\mathbf{f} : W \to \mathbb{R}^n$ be a continuous function on the open set W of $\mathbb{R}^m$ and let C be a compact set contained in W. Then $\mathbf{f}(C)$ is compact.*

Proof. Suppose

$$\mathbf{f}(C) \subset \bigcup_{\lambda \in \Lambda} U_\lambda,$$

where each U_λ is open. It follows that

$$C \subset \bigcup_{\lambda \in \Lambda} \mathbf{f}^{-1}(U_\lambda).$$

Since each $\mathbf{f}^{-1}(U_\lambda)$ is open by the continuity of $\mathbf{f}$, there exist $\lambda_1, \ldots, \lambda_k$ such that

$$C \subset \bigcup_{i=1}^{k} \mathbf{f}^{-1}(U_{\lambda_i})$$

because C is compact. It follows that

$$\mathbf{f}(C) \subset \bigcup_{i=1}^{k} U_{\lambda_i}$$

to complete the proof. □

Knowing that a continuous real-valued function is bounded on a compact set will be a common ingredient in proving theorems about differential equations. In fact, a continuous function assumes its maximum and minimum values on a compact set as the next result establishes.

Proposition 1.3 *Let $f : W \to \mathbb{R}$ be a continuous function on the open set W of $\mathbb{R}^m$ and let C be a compact set contained in W. Then f is bounded on C and there exist $\mathbf{x}_m$ and $\mathbf{x}_M$ in C such that*

$$f(\mathbf{x}_m) \le f(\mathbf{x}) \le f(\mathbf{x}_M)$$

for all $\mathbf{x}$ in C.

Proof. Since $f(C)$ is compact by the previous theorem, it is closed and bounded by the Heine-Borel theorem. Hence, $\inf\{f(\mathbf{x}) : \mathbf{x} \in C\}$ and $\sup\{f(\mathbf{x}) : \mathbf{x} \in C\}$ are finite and belong to $f(C)$. So, there exist $\mathbf{x}_m$ and $\mathbf{x}_M$ in C such that

$$f(\mathbf{x}_m) = \inf\{f(\mathbf{x}) : \mathbf{x} \in C\}$$

and

$$f(\mathbf{x}_M) = \sup\{f(\mathbf{x}) : \mathbf{x} \in C\}.$$

Obviously $f(\mathbf{x}_m) \le f(\mathbf{x}) \le f(\mathbf{x}_M)$ for all $\mathbf{x}$ in C. □

Let $\mathbf{f} : W \to \mathbb{R}^n$ be continuous function on an open set W of $\mathbb{R}^m$ and let E be a subset of W. The function $\mathbf{f}$ is *uniformly continuous on E* if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| < \varepsilon$ whenever $\|\mathbf{y} - \mathbf{x}\| < \delta$ and both $\mathbf{x}$ and $\mathbf{y}$ are in E. Again there is an important connection with compactness.

Proposition 1.4 *Let $\mathbf{f} : W \to \mathbb{R}^n$ be continuous function on an open set W of $\mathbb{R}^m$. If C is a compact set contained in W, then $\mathbf{f}$ is uniformly continuous on C.*

Proof. Let $\varepsilon > 0$. Because $\mathbf{f}$ is continuous at every point of W, it follows that for each $\mathbf{x} \in C$ there exists $\delta_{\mathbf{x}} > 0$ such that $\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| < \varepsilon/2$ when $\|\mathbf{y} - \mathbf{x}\| < \delta_{\mathbf{x}}$. Set $U_{\mathbf{x}} = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < \delta_{\mathbf{x}}/2\}$. Then the sets $U_{\mathbf{x}}$ are a family of open sets such that

$$C \subset \bigcup_{\mathbf{x} \in C} U_{\mathbf{x}}.$$

By the Heine-Borel theorem there exists $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in C such that

$$C \subset \bigcup_{j=1}^{k} U_{\mathbf{x}_j}.$$

Let $\delta = \min\{\delta_{\mathbf{x}_1}/2, \ldots, \delta_{\mathbf{x}_k}/2\}$. Suppose $\|\mathbf{x} - \mathbf{y}\| < \delta$ with $\mathbf{x}$ and $\mathbf{y}$ in C. Then there exists j such that $\|\mathbf{y} - \mathbf{x}_j\| < \delta_{\mathbf{x}_j}/2$. It follows that

$$\|\mathbf{x} - \mathbf{x}_j\| \le \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}_j\| < \delta + \delta_{\mathbf{x}_j}/2 \le \delta_{\mathbf{x}_j}/2 + \delta_{\mathbf{x}_j}/2 = \delta_{\mathbf{x}_j}.$$

By the choice of $\delta_{\mathbf{x}_j}$, both $\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}_j)\| < \varepsilon/2$ and $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_j)\| < \varepsilon/2$. The triangle inequality implies that $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon$ to complete the proof. □

The entire discussion of open, closed, and compact sets originated from the Euclidean distance between two points. Although we are most familiar with

Euclidean distance, it is not, however, the only distance function on which the discussion could have been based. Many arguments will require estimates of distance or size based on norms, and frequently norms other than the Euclidean norm will be easier to apply. To this end, a discussion of norms in general is worth the time, and it will be helpful to prove that it does not matter which norm is used.

A real-valued function $\|\mathbf{x}\|_a$ on $\mathbb{R}^m$ is called a *norm* if it satisfies the following conditions:

(a) $\|\mathbf{x}\|_a = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(b) $\|\alpha\mathbf{x}\|_a = |\alpha|\, \|\mathbf{x}\|_a$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^m$

(c) $\|\mathbf{x}+\mathbf{y}\|_a \leq \|\mathbf{x}\|_a + \|\mathbf{y}\|_a$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$.

As with the Euclidean norm it follows from the triangle inequality that

$$\big|\|\mathbf{x}\|_a - \|\mathbf{y}\|_a\big| \leq \|\mathbf{x}-\mathbf{y}\|_a.$$

Two other simple examples norms on $\mathbb{R}^m$ are

$$|\mathbf{x}| = \sum_{i=1}^{m} |x_i|$$

and

$$\|\mathbf{x}\|_\infty = \max\left\{|x_i| : 1 \leq i \leq m\right\}.$$

The first of these two norms will be particularly useful.

To what extent do the topological ideas of open, closed, and compact sets depend on the norm is now an obvious question. Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$ are called *equivalent* if there exists positive constants A and B satisfying

$$A\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq B\|\mathbf{x}\|_1$$

for all $\mathbf{x}$. [The dot $\cdot$ in the notation $|\cdot|$ or $\mathbf{g}(\cdot)$ indicates an unnamed variable of a norm or a function.]

If two norms are equivalent, then for a given $\mathbf{x}$

$$\left\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\|_1 < r/B\right\} \subset \left\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\|_2 < r\right\} \subset \left\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\|_1 < r/A\right\},$$

and either norm will define the same family of open sets. The next theorem completely settles the question of which norm to use by establishing that they are all equivalent. This fact will be technically very helpful.

Theorem 1.5 *Any two norms on $\mathbb{R}^m$ are equivalent.*

Proof. It suffices to show that the Euclidean norm, $\|\cdot\|$, is equivalent to an arbitrary norm $\|\cdot\|_a$. (See *Exercise 17.*) Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the *standard basis of* $\mathbb{R}^m$, that is, $\mathbf{e}_1 = (1,0,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,0\ldots,0)$, etc. Set

$$c = \max\left\{\|\mathbf{e}_j\|_a : 1 \leq j \leq m\right\}.$$

Then

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j$$

and

$$\|\mathbf{x}\|_a \le \sum_{j=1}^{m} |x_j| \, \|\mathbf{e}_j\|_a \le c \sum_{j=1}^{m} |x_j| \le mc\|\mathbf{x}\|$$

because $|x_j| \le \|\mathbf{x}\|$. This establishes the first required inequalities for the equivalence of norms.

Also note that

$$\big| \|\mathbf{x}\|_a - \|\mathbf{y}\|_a \big| \le \|\mathbf{x} - \mathbf{y}\|_a \le mc\|\mathbf{x} - \mathbf{y}\|$$

implies that $\|\mathbf{x}\|_a$ is a continuous function of $\mathbf{x}$ on $\mathbb{R}^m$. (Given $\varepsilon > 0$, let $\delta = \varepsilon/mc$.)

To establish the second required inequality, it suffices to show that there exists $A > 0$ such that $\|\mathbf{x}\| = 1$ implies $A \le \|\mathbf{x}\|_a$ because then for any $\mathbf{x} \ne \mathbf{0}$,

$$\left\| \tfrac{1}{\|\mathbf{x}\|} \mathbf{x} \right\| = 1$$

and hence

$$A \le \left\| \tfrac{1}{\|\mathbf{x}\|} \mathbf{x} \right\|_a = \frac{1}{\|\mathbf{x}\|} \|\mathbf{x}\|_a$$

or

$$A\|\mathbf{x}\| \le \|\mathbf{x}\|_a .$$

Since $\{\mathbf{x} : \|\mathbf{x}\| = 1\}$ is compact and $\|\mathbf{x}\|_a$ is continuous, by *Proposition 1.3*, there exists $\mathbf{x}_m$ with $\|\mathbf{x}_m\| = 1$ such that

$$\|\mathbf{x}_m\|_a \le \|\mathbf{x}\|_a$$

whenever $\|\mathbf{x}\| = 1$. To complete the proof, set $A = \|\mathbf{x}_m\|_a$, which is positive because $\mathbf{x}_m \ne 0$. □

The above theorem is only true for finite-dimensional vector spaces. In fact, its failure in the infinite-dimensional case is one of the key differences between finite-dimensional and infinite-dimensional normed vector spaces.

Sequences of vector valued functions play a critical role in the study of differential equations, especially uniformly convergent sequences. They are the final preparatory topic in this section. Let $\mathbf{f}_k : W \to \mathbb{R}^n$ be a sequence of continuous functions on an open set $W \subset \mathbb{R}^m$. The sequence of functions $\mathbf{f}_k$ converges to a function $\mathbf{f} : W \to \mathbb{R}^n$ if the sequence $\mathbf{f}_k(\mathbf{x})$ converges to $\mathbf{f}(\mathbf{x})$ for every $\mathbf{x} \in W$. In general, the limit of continuous functions need not be a continuous function unless the convergence is uniform.

The sequence of functions $\mathbf{f}_k$ *converges uniformly on* W to a function $\mathbf{f} : W \to \mathbb{R}^n$ if given $\varepsilon > 0$ there exists $N > 0$ such that $\|\mathbf{f}_k(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| < \varepsilon$ for every $\mathbf{x} \in W$ when $k \ge N$. The proofs of several crucial theorems in this chapter depend on showing that a sequence of functions converges uniformly on an open set and then applying the following result:

Proposition 1.6 *Let $\mathbf{f}_k : W \to \mathbb{R}^n$ be a sequence of continuous functions on an open set $W \subset \mathbb{R}^m$. If $\mathbf{f}_k$ converges uniformly to a function $\mathbf{f} : W \to \mathbb{R}^n$, then $\mathbf{f}$ is continuous on W.*

Proof. Let $\mathbf{y}$ be a point in W. Given $\varepsilon > 0$, there exists $N > 0$ such that $\|\mathbf{f}_k(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| < \varepsilon/3$ for every $\mathbf{x} \in W$ when $k \geq N$. Choose a $k \geq N$. Since $\mathbf{f}_k$ is continuous at $\mathbf{y}$, there exists $\delta > 0$ such that $\|\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{y})\| < \varepsilon/3$ when $\|\mathbf{x} - \mathbf{y}\| < \delta$.

Putting the pieces together

$$\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})\| \leq \|\mathbf{f}(\mathbf{x})-\mathbf{f}_k(\mathbf{x})\|+\|\mathbf{f}_k(\mathbf{x})-\mathbf{f}_k(\mathbf{y})\|+\|\mathbf{f}_k(\mathbf{y})-\mathbf{f}(\mathbf{y})\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

when $\|\mathbf{x} - \mathbf{y}\| < \delta$, and thus $\mathbf{f}$ is continuous at $\mathbf{x}$. □

The proof of the existence of solutions to $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, the first major result about differential equations, will require a theorem known as Ascoli's Lemma. Since it is not as commonly known as other parts of advanced calculus and real analysis, a complete proof of it is included.

Consider a sequence of functions $\mathbf{f}_m : I \to \mathbb{R}^n$ defined on an interval I. The set of functions $\{\mathbf{f}_m : m \geq 1\}$ is *equicontinuous* if given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $m \geq 1$

$$\|\mathbf{f}(s) - \mathbf{f}(t)\| < \varepsilon,$$

whenever $|s - t| < \delta$.

Theorem 1.7 (Ascoli) *Let $\mathbf{f}_m : I \to \mathbb{R}^n$ be a sequence of functions defined on a bounded interval I. If the set of functions $\{\mathbf{f}_m : m \geq 1\}$ is equicontinuous and for each $t \in I$, the sequence $\mathbf{f}_m(t)$ is bounded, then there exists a subsequence of $\mathbf{f}_m$, which converges uniformly on I.*

Proof. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers in I. (See *Exercise 10.*) Since $\mathbf{f}_m(r_1)$ is a bounded sequence of vectors in $\mathbb{R}^n$, it contains a convergent subsequence by the Bolzano-Weierstrass theorem. Thus there exists a subsequence of $\mathbf{f}_m$ denoted by $\mathbf{f}_{(p,1)}$ such that $\mathbf{f}_{(p,1)}(r_1)$ converges to an element of $\mathbb{R}^n$. Since a subsequence of the sequence $\mathbf{f}_m$ is determined by picking an increasing sequence of integer indices, $(p, 1)$ is just the notation being used for the pth integer in the increasing sequence of integers that determines the first subsequence of $\mathbf{f}_m$.

For the same reason, there exists a subsequence $\mathbf{f}_{(p,2)}$ of $\mathbf{f}_{(p,1)}$ such that $\mathbf{f}_{(p,2)}(r_2)$ converges in $\mathbb{R}^n$. Of course, $\lim_{p\to\infty} \mathbf{f}_{(p,2)}(r_1)$ remains unchanged because a subsequence of a convergent sequence converges to the same limit. Now $(p, 2)$ denotes the pth term of the second subsequence. Because we are selecting a subsequence of a subsequence, the increasing sequence of integers $(p, 2)$ must be selected from the increasing sequence $(p, 1)$. Since this process must be repeated ad infinitum, this notation is not as strange as it might first seem, and $\mathbf{f}_{(p,k)}$ will naturally denote the pth in the kth subsequence of $\mathbf{f}_m$.

Using induction, it follows that for every k there exists a subsequence $\mathbf{f}_{(p,k)}$ of $\mathbf{f}_m$ satisfying:

(a) $\mathbf{f}_{(p,k)}$ is a subsequence of $\mathbf{f}_{(p,j)}$ for $j \leq k$, and

(b) $\mathbf{f}_{(p,k)}(r_j)$ converges for $j \leq k$.

Set $\mathbf{g}_p = \mathbf{f}_{(p,p)}$ and verify that $\mathbf{g}_p$ with $p \geq k$ is a subsequence of $\mathbf{f}_{(p,k)}$. In particular, it follows that $\mathbf{g}_p$ is a subsequence of $\mathbf{f}_m$ and that $\mathbf{g}_p(r_k)$ converges for every rational number r_k in I. The proof will be completed by showing that the sequence $\mathbf{g}_p$ is uniformly convergent on I.

Let $\varepsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that for all p

$$\|\mathbf{g}_p(s) - \mathbf{g}_p(t)\| < \frac{\varepsilon}{3},$$

whenever $|s - t| < \delta$, $s, t \in I$. Because the rational numbers are dense in $\mathbb{R}$ and I is bounded, there exists k such that for every $t \in I$ we have $|t - r_i| < \delta$ for some $i \leq k$. Since the sequences $\mathbf{g}_p(r_i)$, $1 \leq i \leq k$ are all Cauchy sequences of real numbers, there exists N such that for $1 \leq i \leq k$

$$\|\mathbf{g}_p(r_i) - \mathbf{g}_q(r_i)\| < \frac{\varepsilon}{3}$$

whenever $p, q \geq N$. Now, let $t \in I$ and consider $p, q \geq N$. Select an r_i, $1 \leq i \leq k$, such that $|t - r_i| < \delta$, and then by the triangle inequality we have

$$\|\mathbf{g}_p(t) - \mathbf{g}_q(t)\| \leq$$

$$\|\mathbf{g}_p(t) - \mathbf{g}_p(r_i)\| + \|\mathbf{g}_p(r_i) - \mathbf{g}_q(r_i)\| + \|\mathbf{g}_q(r_i) - \mathbf{g}_q(t)\| \leq$$

$$\tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

Therefore, $\mathbf{g}_p(t)$ is a Cauchy sequence, and hence converges to some $\mathbf{g}(t)$ for all t in I. Letting p go to infinity in the above inequality, gives $\|\mathbf{g}(t) - \mathbf{g}_q(t)\| \leq \varepsilon$ for $t \in I$ and $q \geq N$, and proves that $\mathbf{g}_p(t)$ converges uniformly to $\mathbf{g}(t)$ on I. □

EXERCISES

1. Prove the *Cauchy-Schwarz inequality*:

$$|\mathbf{v}\cdot\mathbf{w}| \leq \|\mathbf{v}\|\,\|\mathbf{w}\|$$

Hint: For fixed $\mathbf{v}$ and $\mathbf{w}$ the function $\|t\mathbf{v}-\mathbf{w}\|^2$ is a quadratic in t whose discriminant must be less than or equal to 0 because $0 \leq \|t\mathbf{v}-\mathbf{w}\|^2$. (The *discriminant* of at^2+bt+c is b^2-4ac.)

2. Use the Cauchy-Schwarz inequality to show that

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

3. Show that for fixed $\mathbf{v}$ and $r > 0$ the set

$$\{\mathbf{x} : \|\mathbf{x}-\mathbf{v}\| < r\}$$

is an open set and the set

$$\{\mathbf{x} : \|\mathbf{x}-\mathbf{v}\| \leq r\}$$

is a closed set.

4. Prove that the union of open sets is open and the intersection of a finite number of open sets is open.

5. Prove that the intersection of closed sets is closed and the finite union of closed sets is closed.

6. Prove that a set C is a closed set if and only if for every convergent sequence $\mathbf{x}_n$ in C, its limit is also in C.

7. Let B be a subset of $\mathbb{R}^m$. Define the *closure* of B by

$$\overline{B} = \{\mathbf{x} : B \cap \{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\| < \varepsilon\} \neq \phi \text{ for all } \varepsilon > 0\}.$$

Prove the following:

   (a) The set $\overline{B}$ is a closed set containing B.
   (b) If C is a closed set such that $B \subset C$, then $\overline{B} \subset C$.
   (c) The set B is closed if and only if $B = \overline{B}$.
   (d) $\overline{B} = \cap\{C : B \subset C = \overline{C}\}$.
   (e) The closure of an open Euclidean ball is a closed Euclidean ball.

8. Let D be an open subset of $\mathbb{R}^d$ and let C be a compact set contained in D. Set $\rho = \inf\{|\mathbf{x}-\mathbf{y}| : \mathbf{x} \in C \text{ and } \mathbf{y} \notin D\}$.

   (a) Show that ρ is positive. Is this true if C is just closed?

(b) Show that $\{\mathbf{y} : |\mathbf{x} - \mathbf{y}| \leq \rho/2 \text{ for some } \mathbf{x} \in C\}$ is a compact set contained in D.

(c) Show that $\{\mathbf{y} : |\mathbf{x} - \mathbf{y}| < \rho/2 \text{ for some } \mathbf{x} \in C\}$ is an open set.

9. A sequence of points $\mathbf{x}_k$ in $\mathbb{R}^m$ is a Cauchy sequence if given $\varepsilon > 0$, there exists $N > 0$ such that $\|\mathbf{x}_j - \mathbf{x}_k\| < \varepsilon$ when $j \geq N$ and $k \geq N$. Using the fact that a sequence of real numbers converges to a real number if and only if it is a Cauchy sequence, show that a sequence $\mathbf{x}_k$ in $\mathbb{R}^m$ converges to a point $\mathbf{y}$ in $\mathbb{R}^m$ if and only if it is Cauchy sequence.

10. Construct a sequence of rational numbers r_n that includes every positive rational number. Modify the construction to include every rational number. Given an interval I, construct a sequence of rational numbers r_n that includes every rational number in the interval I.

11. A set B is *countable* provided there exists a sequence $\mathbf{x}_k$ such that $\mathbf{x}$ is in B if and only if $\mathbf{x} = \mathbf{x}_k$ for some k. The rational numbers are countable by the previous exercise. Prove that the set

$$\{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_j \text{ is rational for } j = 1, 2, \ldots, m\}$$

is countable.

12. Prove that every open set of $\mathbb{R}^m$ is a union of Euclidean balls of the form $\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < r\}$, where r is rational and the center $\mathbf{x}$ has rational coordinates.

13. Let $\mathbf{f} : \Omega \to \mathbb{R}^n$ be a function, where Ω is an open subset of $\mathbb{R}^m$. Prove that f is continuous at every point of Ω if and only if $f^{-1}(U) = \{\mathbf{x} : f(\mathbf{x}) \in U\}$ is an open set for every open set U of $\mathbb{R}^n$.

14. Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a function. Prove that f is continuous at every point of $\mathbb{R}^m$ if and only if $f^{-1}(C) = \{\mathbf{x} : f(\mathbf{x}) \in C\}$ is a closed set for every closed set C of $\mathbb{R}^n$.

15. Let $\mathbf{f} : \Omega \to \mathbb{R}^n$ be a function, where Ω is an open subset of $\mathbb{R}^m$. Prove that f is continuous at $\mathbf{x} \in \Omega$ if and only if for every sequence $\{\mathbf{x}_k\}$ converging to $\mathbf{x}$, the sequence $\{\mathbf{f}(\mathbf{x}_k)\}$ converges to $\mathbf{f}(\mathbf{x})$.

16. Show that

$$|\mathbf{x}| = \sum_{i=1}^{d} |x_i|$$

and

$$\|\mathbf{x}\|_\infty = \max\{|x_i| : 1 \leq i \leq d\}$$

define norms on $\mathbb{R}^d$ and find A and B such that $A|\mathbf{x}| \leq \|\mathbf{x}\|_\infty \leq B|\mathbf{x}|$. For $d = 2$, graph the sets $\{\mathbf{x} : |\mathbf{x}| \leq 1\}$ and $\{\mathbf{x} : \|\mathbf{x}\|_\infty \leq 1\}$.

17. Let $\|\cdot\|_a$, $\|\cdot\|_b$, and $\|\cdot\|_c$ be three norms on a vector space V. Show that if $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms and if $\|\cdot\|_b$ and $\|\cdot\|_c$ are equivalent norms, then $\|\cdot\|_a$ and $\|\cdot\|_c$ are also equivalent norms.

18. Show by example that Ascoli's Lemma is false when each of the following hypotheses are individually deleted from the statement of the result:

    (a) The interval I is bounded,

    (b) The sequence of functions $\{f_m\}$ is equicontinuous,

    (c) The sequence $\{f_m(t)\}$ is bounded for every $t \in I$.

1.2 Existence

The study of differential equations can now begin in earnest. The class of differential equations to be examined needs to be fully described, the concept of a solution of a differential equation needs to be defined, and the question of whether or not a particular differential equation has solutions needs to be addressed. These things will take place in this section and launch all that follows.

Let D be an open set of $\mathbb{R}^{d+1}$ and let $\mathbf{f} : D \to \mathbb{R}^d$ be a continuous function. As is customary in differential equations, a point of $\mathbb{R}^{d+1}$ will be denoted by $(t, \mathbf{x})$, where $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$. Moreover, in this notation t will be thought of as time and $\mathbf{x}$ as position in space. In this context, the *differential equation*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{1.1}$$

is the most general differential equation that will be considered, and the notation set out in this paragraph will be used consistently for it.

Actually, equation *(1.1)* is a system of differential equations. Since

$$\mathbf{f}(t, \mathbf{x}) = \big(f_1(t, \mathbf{x}), \ldots, f_d(t, \mathbf{x})\big)$$

where the f_i are continuous real-valued functions, $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ can also be written

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, x_2, \ldots, x_d) \\
\dot{x}_2 &= f_2(t, x_1, x_2, \ldots, x_d) \\
\vdots \quad & \quad \vdots \\
\dot{x}_d &= f_d(t, x_1, x_2, \ldots, x_d).
\end{aligned}$$

Let I be an open interval (possibly infinite). A *curve* is a continuous function $\boldsymbol{\varphi} : I \to \mathbb{R}^d$. The curve $\boldsymbol{\varphi} : I \to \mathbb{R}^d$ is said to be differentiable if $\dot{\boldsymbol{\varphi}} = (\dot{\varphi}_1, \ldots, \dot{\varphi}_d)$ exists at every point of I, where as usual $\dot{\varphi}_i$ is the derivative of φ_i, which is a real-valued function of one real variable. A *solution* of 1.1 is simply a differentiable curve $\boldsymbol{\varphi} : I \to \mathbb{R}^d$ such that on I

$$\dot{\boldsymbol{\varphi}}(t) = \mathbf{f}\big(t, \boldsymbol{\varphi}(t)\big).$$

Note it follows that if φ is a solution of *(1.1)*, then $(t, \varphi(t))$ is in D for $t \in I$, and the curve $t \to (t, \varphi(t))$ lying in D is called a *trajectory* of *(1.1)*. Because $\mathbf{f}$ and φ are continuous, it also follows that $\dot{\varphi}$ is continuous. That is, a solution must be continuously differentiable. Instead of using a different symbol for the curve φ and the point $\mathbf{x}$, we will usually write $\mathbf{x}(t)$ to denote dependency of $\mathbf{x}$ on t in a solution of *(1.1)*.

For differential equations, it is slightly more convenient to use the norm

$$|\mathbf{x}| = \sum_{i=1}^{d} |x_i|$$

and define the distance between $\mathbf{x}$ and $\mathbf{y}$ to be $|\mathbf{x} - \mathbf{y}|$. Using $|\mathbf{x}|$ instead of the more familiar $\|\mathbf{x}\|$ has no effect on the open sets, the continuity of functions, and the convergence of sequences by *Theorem 1.5*. Not only are $\|\cdot\|$ and $|\cdot|$ equivalent, it can easily be verified that

$$\frac{1}{d}|\mathbf{x}| \leq \|\mathbf{x}\| \leq |\mathbf{x}|.$$

Let φ be a curve whose domain includes the closed interval $[a, b]$. Define the integral of φ from a to b by

$$\int_a^b \varphi(t)\, dt = \left(\int_a^b \varphi_1(t)\, dt, \ldots, \int_a^b \varphi_d(t)\, dt \right).$$

When $a < b$, then

$$\begin{aligned} \left| \int_a^b \varphi(t)\, dt \right| &= \sum_{i=1}^{d} \left| \int_a^b \varphi_i(t)\, dt \right| \\ &\leq \sum_{i=1}^{d} \int_a^b |\varphi_i(t)|\, dt \\ &= \int_a^b \sum_{i=1}^{d} |\varphi_i(t)|\, dt \\ &= \int_a^b |\varphi(t)|\, dt. \end{aligned}$$

So in this context, the familiar inequality

$$\left| \int_a^b \varphi(t)\, dt \right| \leq \int_a^b |\varphi(t)|\, dt$$

when $a < b$ is retained and is one reason for preferring the norm $|\cdot|$ to the Euclidean norm. The next remark shows why the integral of a curve is relevant for differential equations.

Proposition 1.8 *Let* $\mathbf{x}(t)$ *be a continuous function on the open interval* I *such that* $(t, \mathbf{x}(t)) \in D$ *for all* $t \in I$, *and let* $\tau \in I$. *Then* $\mathbf{x}(t)$ *is a solution of* $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ *if and only if*

$$\mathbf{x}(t) = \mathbf{x}(\tau) + \int_\tau^t \mathbf{f}(s, \mathbf{x}(s))\, ds. \tag{1.2}$$

Proof. Just apply the fundamental theorem of calculus to each coordinate. □

The first fundamental question about the differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ is whether or not any solutions exist, but this question cannot be phrased quite so simply. Since the trajectory of one solution occupies an insignificant portion of D, knowing that *(1.1)* has a solution tells us very little. Furthermore, in many physical situations, the initial data specifies a point in D through which the desired trajectory must pass.

The right existence question to ask is When does there exist a trajectory passing through a specific point of D? The answer to this question is the best possible, and the primary goal of this section is to prove that at least one trajectory passes through each point of D.

Seeking a solution whose trajectory passes through a specified point $(\tau, \boldsymbol{\xi}) \in D$ is called an initial-value problem. Note that $(\tau, \boldsymbol{\xi})$ is on the trajectory of the solution $\mathbf{x}(t)$ if and only if $(\tau, \mathbf{x}(\tau)) = (\tau, \boldsymbol{\xi})$ or $\mathbf{x}(\tau) = \boldsymbol{\xi}$. Consequently, an *initial-value problem* can always be written in the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau). \end{aligned}$$

A solution to the above initial-value problem starts at $\boldsymbol{\xi}$ at time τ and heads in the $\mathbf{f}(\tau, \boldsymbol{\xi})$ direction for just an instant. By actually following the line through $\boldsymbol{\xi}$ in the $\mathbf{f}(\tau, \boldsymbol{\xi})$ direction for a short time we will not stay on the solution, but the error should be small over a short time interval. We can also stop and correct our course by recalculating the direction $\mathbf{f}(t, \mathbf{x})$ at a new time and point, and then follow the line in the corrected direction. Repeating this process for short intervals of time should track an actual solution. The challenge is to show that this intuitive idea really works.

Theorem 1.9 (Peano) *If* $\mathbf{f}(t, \mathbf{x})$ *is continuous on the open set* D, *then for each point* $(\tau, \boldsymbol{\xi}) \in D$, *there exists at least one solution to the initial-value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau). \end{aligned} \tag{1.3}$$

Before tackling the details of the proof, it is worthwhile outlining the argument to understand how the pieces will fit together. The first step is to define a sequence of approximate solutions

$$\varphi_m : \{t : \tau - \alpha < t < \tau + \alpha\} \to \mathbb{R}^d$$

for a suitable $\alpha > 0$ using line segments. The function $\mathbf{f}(t, \mathbf{x})$ will be used to determine the directions of these line segments. Specifically, starting at the point $\boldsymbol{\xi}$ and at time τ, the curve $\varphi_m(t)$ will move along the line through $\boldsymbol{\xi}$ in the direction $\mathbf{f}(\tau, \boldsymbol{\xi})$ until a time t_1. From t_1 to t_2, it will move along the line through $\varphi_m(t_1)$ in the direction $\mathbf{f}(t_1, \varphi_m(t_1))$, and so forth. Of course, as m gets large, the distance between τ and t_1, t_1 and t_2, and so forth, will be sent to zero.

The second step is to show that Ascoli's lemma applies to the sequence φ_m of functions, and thus establishes the existence of a subsequence φ_{m_k}, which converges uniformly on $[\tau - \alpha, \tau + \alpha]$ to a function φ.

The final step is to prove that φ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ satisfying $\varphi(\tau) = \boldsymbol{\xi}$. Carrying out the details of the proof will take several pages and may take more than one reading to understand completely.

Proof. Because D is open, there exists $b > 0$ such that for the given point $(\tau, \boldsymbol{\xi}) \in D$, the set

$$R = \{(t, \mathbf{x}) : |t - \tau| \le b \text{ and } |\mathbf{x} - \boldsymbol{\xi}| \le b\} \subset D.$$

Since the rectangle R is closed and bounded and hence compact, it follows from *Proposition 1.3* that

$$\sup\{|\mathbf{f}(t, \mathbf{x})| : (t, \mathbf{x}) \in R\} = M < \infty.$$

Set $\alpha = \min\{b, b/M\}$ and let I denote the open interval $\{t : \tau - \alpha < t < \tau + \alpha\}$.

For each positive integer m, an approximate solution $\varphi_m : I \to \mathbb{R}^n$ will be defined. Because R is compact, $\mathbf{f}$ is uniformly continuous on R by *Proposition 1.4*. In particular, there exists $\delta_m > 0$ such that

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(s, \mathbf{y})| \le \frac{1}{m},$$

whenever $|t - s| < \delta_m$ and $|\mathbf{x} - \mathbf{y}| < \delta_m$. Now, fix m and pick t_i such that

$$\tau - \alpha = t_{-p} < t_{-p+1} < \cdots < t_{-1} < t_0 = \tau < t_1 < \cdots < t_{p-1} < t_p = \tau + \alpha$$

and $|t_i - t_{i-1}| < \min\{\delta_m, \delta_m/M, 1/m\}$. Define $\varphi_m(t)$ on the closed interval $[t_{-1}, t_1]$ by

$$\begin{aligned} \varphi_m(t) &= \boldsymbol{\xi} + (t - t_0) \cdot \mathbf{f}(t_0, \boldsymbol{\xi}) \\ &= \boldsymbol{\xi} + (t - \tau) \cdot \mathbf{f}(\tau, \boldsymbol{\xi}). \end{aligned}$$

Note that this is a line segment with $\varphi_m(t_0) = \varphi_m(\tau) = \boldsymbol{\xi}$. For $t \in [t_{-1}, t_1]$,

$$|\varphi_m(t) - \boldsymbol{\xi}| = |t - t_0|\,|\mathbf{f}(t_0, \boldsymbol{\xi})| \le \alpha M \le \frac{b}{M} M = b$$

and $(t, \varphi_m(t)) \in R$. Extend the definition of φ_m to $[t_1, t_2]$ by

$$\varphi_m(t) = \varphi_m(t_1) + (t - t_1)\,\mathbf{f}(t_1, \varphi_m(t_1)).$$

For $t \in [t_1, t_2]$,

$$\begin{aligned} |\varphi_m(t) - \boldsymbol{\xi}| &\leq |\varphi_m(t) - \varphi_m(t_1)| + |\varphi_m(t_1) - \boldsymbol{\xi}| \\ &\leq |t - t_1| \, |\mathbf{f}(t_1, \varphi_m(t_1))| + |t_1 - t_0| \, |\mathbf{f}(t_0, \boldsymbol{\xi})| \\ &\leq |t - t_1| M + |t_1 - t_0| M = |t - t_0| M \leq b \end{aligned}$$

and $(t, \varphi_m(t)) \in R$ for $t_0 \leq t \leq t_2$. It is now clear that this process can be repeated in both directions until it reaches $t_p = \tau + \alpha$ and $t_{-p} = \tau - \alpha$ to define φ_m on I satisfying $(t, \varphi_m(t)) \in R$ for all $t \in I$. In particular,

$$\varphi_m(t) = \varphi_m(t_k) + (t - t_k)\, \mathbf{f}(t_k, \varphi_m(t_k))$$

on $[t_k, t_{k+1}]$ for $k \geq 0$ and on $[t_{k-1}, t_k]$ for $k \leq 0$. See *Figure 1.1* for an illustration of a $\varphi_m(t)$ in the plane.

Clearly, φ_m is continuous and

$$\dot{\varphi}(t) = \mathbf{f}(t_k, \varphi_m(t_k))$$

when $t_k < t < t_{k+1}$ for $k \geq 0$ and when $t_{k-1} < t < t_k$ for $k \leq 0$. This completes the first step in the proof.

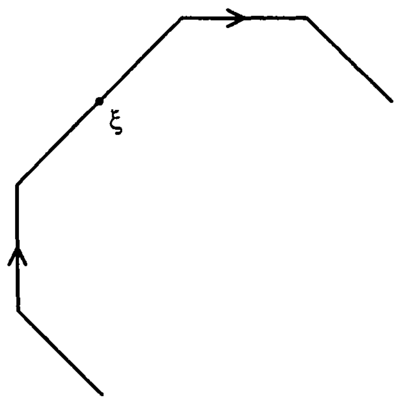


Figure 1.1: Graph of a sample planar $\varphi_m(t)$ with $p = 3$. The arrows indicate the direction of increasing t.

The crucial property of φ_m that must be established is the following: For s and t in I

$$|\varphi_m(t) - \varphi_m(s)| < |t - s| M.$$

The case when $t_0 \leq s < t$ will be established and the other two cases will be left to the reader. First, find j and k, $0 \leq j \leq k$ such that $t_j \leq s < t_{j+1}$ and $t_k \leq t < t_{k+1}$. If $j = k$, then

$$|\varphi_m(t) - \varphi_m(s)| = |t - s| |\mathbf{f}(t_j, \varphi_m(t_j))| \leq |t - s| M.$$

If $j < k$, then

$$\begin{aligned}
&\left|\varphi_m(t) - \varphi_m(s)\right| \leq \\
&\left|\varphi_m(t) - \varphi_m(t_k)\right| + \left|\varphi_m(t_k) - \varphi_m(t_{k-1})\right| + \cdots + \left|\varphi_m(t_{j+1}) - \varphi_m(s)\right| \leq \\
&|t - t_k| \left|\mathbf{f}\big(t_k, \varphi_m(t_k)\big)\right| + \cdots + |t_{j+1} - s| \left|\mathbf{f}\big(t_j, \varphi_m(t_j)\big)\right| \leq \\
&|t - t_k|M + |t_k - t_{k-1}|M + \cdots + |t_{j+1} - s|M = \\
&|t - s|M.
\end{aligned}$$

(The last equality holds because $|t - t_k| = t - t_k$, etc.) The proof of the other two cases is similar.

For $t \neq t_i$, $i = -p, \ldots, p$, we have $\dot{\varphi}_m(t) = \mathbf{f}\big(t_k, \varphi_m(t_k)\big)$ for some t_k such that $|t - t_k| < \delta_m$ and $|t - t_k| < \delta_m/M$. Hence $\left|\varphi_m(t) - \varphi_m(t_k)\right| < |t - t_k|M < \delta_m$, and the uniform continuity of $\mathbf{f}$ now implies that

$$\left|\dot{\varphi}_m(t) - \mathbf{f}\big(t, \varphi_m(t)\big)\right| = \left|\mathbf{f}\big(t_k, \varphi_m(t_k)\big) - \mathbf{f}\big(t, \varphi_m(t)\big)\right| < \frac{1}{m},$$

which will be needed for the last step. The above inequality also says that each φ_m is an *approximate solution* of *(1.3)*.

Since $|t - s|M$ is independent of m, it follows from $\left|\varphi_m(t) - \varphi_m(s)\right| \leq |t - s|M$ that $\{\varphi_m : m \geq 1\}$ is an equicontinuous set of functions. Furthermore, φ_m is a uniformly bounded sequence because

$$\begin{aligned}
\left|\varphi_m(t)\right| &\leq \left|\varphi_m(t) - \varphi_m(t_0)\right| + \left|\varphi_m(t_0)\right| \\
&\leq |t - t_0|M + \left|\varphi_m(t_0)\right| \\
&\leq \alpha M + |\boldsymbol{\xi}|.
\end{aligned}$$

Hence, Ascoli's Lemma *Theorem 1.7* applies and there exists a subsequence φ_{m_i}, which is uniformly convergent on I to some function φ. This completes the second step of the proof.

Clearly, $\varphi_m(\tau) = \boldsymbol{\xi}$ for all m implies $\varphi(\tau) = \boldsymbol{\xi}$. To establish that φ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ and complete the proof, it suffices by *Proposition 1.8* to show that *(1.2)* holds or equivalently that

$$\left|\varphi(t) - \varphi(\tau) - \int_\tau^t \mathbf{f}\big(s, \varphi(s)\big)\, ds\right| = 0$$

on I.

By the triangle inequality,

$$\begin{aligned}
&\left|\varphi(t) - \varphi(\tau) - \int_\tau^t \mathbf{f}\big(s, \varphi(s)\big)\, ds\right| \\
&\qquad \leq \left|\varphi(t) - \varphi_{m_i}(t)\right| + \left|\varphi_{m_i}(t) - \varphi(\tau) - \int_\tau^t \mathbf{f}\big(s, \varphi_{m_i}(s)\big)\, ds\right| \\
&\qquad\quad + \left|\int_\tau^t \mathbf{f}\big(s, \varphi_{m_i}(s)\big) - \mathbf{f}\big(s, \varphi(s)\big)\, ds\right|.
\end{aligned}$$

The first term obviously goes to zero as i goes to infinity. With the observations that

$$\left|\int_\tau^t \mathbf{f}(s,\varphi_{m_i}(s)) - \mathbf{f}(s,\varphi(s))\,ds\right| \le \int_\tau^t \left|\mathbf{f}(s,\varphi_{m_i}(s)) - \mathbf{f}(s,\varphi(s))\right| ds$$

and

$$\left|\mathbf{f}(s,\varphi_{m_i}(s)) - \mathbf{f}(s,\varphi(s))\right| \to 0$$

uniformly as i goes to infinity, it follows that the third term also goes to zero as i goes to infinity.

Because $\dot{\varphi}_m$ is piecewise continuous, it follows that

$$\varphi_m(t) = \varphi_m(\tau) + \int_\tau^t \dot{\varphi}_m(s)\,ds$$

for all m. Now substitute the above expression into the middle term to get

$$\begin{aligned}\left|\int_\tau^t \dot{\varphi}_{m_i}(s) - \mathbf{f}(s,\varphi_{m_i}(s))\,ds\right| &\le \int_\tau^t \left|\dot{\varphi}_{m_i}(s) - \mathbf{f}(s,\varphi_{m_i}(s))\right| ds \\ &\le \int_\tau^t \frac{1}{m_i}\,ds \le \frac{\alpha}{m_i}\end{aligned}$$

because φ_{m_i} is an approximate solution. Thus, the middle term goes to zero as i goes to infinity to complete the proof. □

Corollary *If* $\mathbf{f}$ *is continuous on an open set* D*, then every point in* D *has at least one trajectory of* $\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x})$ *passing through it.*

Corollary *If* $\mathbf{f}$ *is continuous on an open set* D *and* C *is a compact subset of* D*, then there exists* $\alpha > 0$ *such that for every* $(t,\boldsymbol{\xi}) \in C$*, the initial-value problem*

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau)\end{aligned}$$

has a solution defined on $\{t : \tau = \alpha < t < \tau + \alpha\}$.

Proof. Exercise.

Peano's theorem supplies a simple general setting in which the existence of solutions can be taken for granted. We will always stay within this context and only consider the differential equations $\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x})$, where $\mathbf{f}$ is continuous on an open set.

Peano's theorem, however, has three drawbacks. First, the interval on which the solution is defined in the proof of Peano's theorem may unnecessarily be very short. Second, it leaves open the possibility that an initial-value problem has more than one solution. Third, it is highly non constructive.

EXERCISES

1. Consider the initial-value problem

$$\begin{aligned}\dot{x} &= t^2 + x^2 \\ 0 &= x(0)\end{aligned}$$

on $\mathbb{R}^2$. Determine the longest interval on which the proof of Peano's theorem guarantees a solution. What is the answer to the same question when $t^2 + x^2$ is replaced by $|t|^p + |x|^p$ with $p > 1$?

2. Let $D = \mathbb{R}^2$ and use separation of variables to solve the initial-value problem

$$\begin{aligned}\dot{x} &= 1 + x^2 \\ 0 &= x(0).\end{aligned}$$

(In this problem, $f(t, x) = 1 + x^2$ is independent of t.) Show that the longest interval on which Peano's theorem guarantees a solution is less than one-third of the length of the interval on which there is a known solution.

3. Suppose $\mathbf{f}(t, \mathbf{x})$ is continuous and bounded on $\mathbb{R}^d$. Show that for all $\boldsymbol{\xi}$ the initial-value problem

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau)\end{aligned}$$

has a solution defined on an arbitrarily long interval.

4. Prove the second corollary to Peano's theorem.

5. Let $\varphi_m(t)$ be a sequence of solutions to $\dot{x} = \mathbf{f}(t, \mathbf{x})$ with $\mathbf{f}$ continuous on an open set D. Suppose that

$$R = \{(t, \mathbf{x}) : |t - \tau| \leq c \text{ and } |\mathbf{x} - \boldsymbol{\xi}| \leq c\} \subset D$$

for some $c > 0$. Prove the following: If each $\varphi_m(t)$ is defined on the open interval $I = (\tau - \gamma, \tau + \gamma)$, the sequence $\varphi_m(\tau)$ converges to $\boldsymbol{\xi}$, and $(t, \varphi_m(t)) \in R$ for all $t \in I$, then some subsequence of φ_m converges to a solution of

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau).\end{aligned}$$

6. Suppose $\mathbf{f}(t, \mathbf{x})$ is continuous and satisfies $|f(t, \mathbf{x})| \leq \log(|t| + |\mathbf{x}|)$ on $\mathbb{R}^d$. Show that for all $\boldsymbol{\xi}$ the initial-value problem

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau)\end{aligned}$$

has a solution defined on an arbitrarily long interval.

1.3 Uniqueness

The second fundamental question concerning initial-value problems is to determine when there is exactly one solution. This question is not purely question; it has practical consequences.

For example, suppose the behavior of some physical phenomenon is modelled by the initial-value problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \boldsymbol{\xi} &= \mathbf{x}(\tau) \end{aligned}$$

Also suppose a specific function $\varphi(t)$ is a solution of this problem. Without knowing that this initial-value problem has a unique solution it cannot be asserted a priori that $\varphi(t)$ describes the behavior of the physical phenomenon being studied. Its behavior may in fact, be governed by a different solution $\psi(t)$. Consequently, simple broadly applicable tests for uniqueness can guarantee that a known solution of an initial-value problem is not extraneous.

Similarly, if a numerical procedure carried out with the aid of a computer is generating a sequence of approximate solutions to an initial-value problem, then without uniqueness the meaning of these computer calculations can be very ambiguous. In fact, uniqueness ensures that solutions can be approximated by numerical procedures. The most elementary of them, Euler's method, will be discussed in the next section.

Continuity of $\mathbf{f}$ is not enough to guarantee that initial-value problems have unique solutions. To construct a counterexample let $D = \mathbb{R}^2$, that is, $d = 1$ and both t, $x \in \mathbb{R}$.

Consider the initial-value problem

$$\begin{aligned} \dot{x} &= \sqrt{|x|} \\ 0 &= x(0). \end{aligned}$$

[Although $f(t, x)$ depends only on $x \in \mathbb{R}$, the t variable is included in the domain so that for now all differential equations are treated consistently in one form.] Clearly, $x(t) \equiv 0$ or $x(t) = 0$ for all $t \in \mathbb{R}$ is a solution. (The symbol $\equiv$ is used to indicate that a function has a particular constant value at each point in its domain.)

The elementary technique of separating variables yields a second solution,

$$x(t) = \begin{cases} 0, & t \leq 0; \\ t^2/4, & t \geq 0, \end{cases}$$

of the same initial-value problem.

The bad behavior of the above example is a result of the cusp of $\sqrt{|x|}$ at 0. However, there are differential equations that exhibit even more pathological behavior. For example, it is possible to construct a continuous function $f(t, x)$ on $\mathbb{R}^2$ such that every initial-value problem has infinitely many solutions. (See [17] page 18.)

The theme of this section is the study of a simple assumption called a Lipschitz condition, which guarantees that initial-value problems have unique solutions. After the Lipschitz condition is defined and its relationship with uniqueness analyzed, it will be shown that continuous first partial derivatives imply the Lipschitz condition. Thus the usually easy to check hypothesis that a function has continuous first partial derivatives will resolve one of the weaknesses of Peano's theorem.

If there exists a constant $L > 0$ such that for every $(t, \mathbf{x})$ and $(t, \mathbf{y})$ in D the following inequality holds:

$$\left|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})\right| \leq L|\mathbf{x} - \mathbf{y}|, \tag{1.4}$$

$\mathbf{f}(t, \mathbf{x})$ is said to satisfy a *Lipschitz condition* on D. Technically, this Lipschitz condition is only with respect to the space variable $\mathbf{x}$. Because no other variations of this concept will be used, it will be adequate simply to say that $\mathbf{f}(t, \mathbf{x})$ satisfies a Lipschitz condition and not include the phrase "with respect to $\mathbf{x}$."

Roughly speaking, a Lipschitz condition says that the values of $\mathbf{f}$ cannot separate faster than the distance between $\mathbf{x}$ and $\mathbf{y}$. It can also be thought of as a crude finite derivative condition because

$$\frac{\left|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})\right|}{|\mathbf{x} - \mathbf{y}|} \leq L$$

when $\mathbf{x} \neq \mathbf{y}$. Replacing $|\cdot|$ by an equivalent norm will not destroy a Lipschitz condition but will change the constant L.

Theorem 1.10 *Let* $\mathbf{f}$ *satisfy a Lipschitz condition on* D *and let* $\varphi_1(t)$ *and* $\varphi_2(t)$ *be two solutions of the differential equation* $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ *on the domain* D. *Suppose both* $\varphi_1(t)$ *and* $\varphi_2(t)$ *are defined on the open interval* I. *If* $\varphi_1(\tau) = \varphi_2(\tau)$ *for some* $\tau \in I$, *then* $\varphi_1(t) = \varphi_2(t)$ *for every* t *in* I.

Proof. Because $\varphi_1(\tau) = \varphi_2(\tau)$, it follows from

$$\varphi_i(t) = \varphi_i(\tau) + \int_\tau^t \mathbf{f}(s, \varphi_i(s))ds,$$

and *(1.4)* that

$$\begin{aligned}
\left|\varphi_1(t) - \varphi_2(t)\right| &= \left|\int_\tau^t \mathbf{f}\big(s,\varphi_1(s)\big) - \mathbf{f}\big(s,\varphi_2(s)\big)\, ds\right| \\
&\leq \int_\tau^t \left|\mathbf{f}\big(s,\varphi_1(s)\big) - \mathbf{f}\big(s,\varphi_2(s)\big)\right| ds \\
&\leq \int_\tau^t L\left|\varphi_1(s) - \varphi_2(s)\right| ds
\end{aligned}$$

for $t \geq \tau$. Thus

$$\left|\varphi_1(t) - \varphi_2(t)\right| \leq L\int_\tau^t \left|\varphi_1(s) - \varphi_2(s)\right| ds$$

for $t \geq \tau$. It remains to show that this inequality forces $\varphi_1(t) = \varphi_2(t)$ for $t \geq \tau$, and then apply the simple technique of reversing time to obtain the result for $t \leq \tau$.

Setting $g(t) = |\varphi_1(t) - \varphi_2(t)|$ and $G(t) = \int_\tau^t g(s)\,ds$, the previous inequality can be written as

$$g(t) \leq LG(t)$$

or

$$\dot{G}(t) - LG(t) \leq 0.$$

Multiply this inequality by e^{-Lt} and note that the left-hand side is now the derivative of $e^{-Lt}G(t)$. (This is the standard method of integrating factors used to solve a first-order linear differential equations.) Thus

$$\frac{d\left[e^{-Lt}G(t)\right]}{dt} \leq 0.$$

for $t \geq \tau$. Because $e^{-L\tau}G(\tau) = 0$, integrating from τ to t, yields

$$e^{-Lt}G(t) \leq 0$$

or

$$G(t) \leq 0$$

when $t \geq \tau$.

Since $\dot{G}(t) = g(t) \geq 0$ and $G(\tau) = 0$, it also follows by integrating $\dot{G}(t)$ that $G(t) \geq 0$ for $t \geq \tau$. Therefore, $G(t) \equiv 0$ and $\dot{G}(t) = g(t) \equiv 0$ for $t \geq \tau$. It follows that $\varphi_1(t) = \varphi_2(t)$ for $t \geq \tau$.

For $t \leq \tau$ and $i = 1, 2$, let $\psi_i(t) = \varphi_i(-t)$ on $-I = \{-t : t \in I\}$. Then

$$\dot{\psi}_i(t) = -\dot{\varphi}_i(-t) = -\mathbf{f}\big(-t, \varphi_i(-t)\big) = -\mathbf{f}\big(-t, \psi_i(t)\big)$$

and $\psi_i(t)$ is a solution of $\dot{\mathbf{x}} = -\mathbf{f}(-t, \mathbf{x})$ on $D' = \{(t, \mathbf{x}) : (-t, \mathbf{x}) \in D\}$. Clearly, $\psi_1(-\tau) = \psi_2(-\tau)$ and $|-\mathbf{f}(-t, \mathbf{x}) - -\mathbf{f}(-t, \mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|$. The preceding arguments show that $\psi_1(s) = \psi_2(s)$ for $s \geq -\tau$ or $\varphi_1(t) = \varphi_2(t)$ for $t \leq \tau$. □

It is not evident from the previous theorem that uniqueness is really a local issue. By "local" we mean what occurs just near points in D not necessarily throughout D. The next theorem makes this point.

Theorem 1.11 *Let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ on the domain D and assume they are defined on the same interval I. Suppose that for each point $\mathbf{x}$ of D there exists an open set U containing $\mathbf{x}$ and contained in D such that $\mathbf{f}(t, \mathbf{x})$ satisfies a Lipschitz condition on U. If $\varphi_1(\tau) = \varphi_2(\tau)$ for some $\tau \in I$, then $\varphi_1(t) = \varphi_2(t)$ for all t in I.*

Proof. Assume $\varphi_1(s) \neq \varphi_2(s)$ for some $s > \tau$. (The argument for the case when $s < \tau$ is similar.) Set

$$\beta = \sup\left\{s : \varphi_1(t) = \varphi_2(t) \text{ for } \tau \leq t \leq s\right\}.$$