THE THEORY OF MEASURES AND INTEGRATION

Eric M. Vestrup

WILEY-INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

This Page Intentionally Left Blank

THE THEORY OF MEASURES AND INTEGRATION

WILEY SERIES IN PROBABILITY AND STATISTICS

Established by WALTER A. SHEWHART and SAMUEL S. WILKS

A complete list of the titles in this series appears at the end of this volume.

THE THEORY OF MEASURES AND INTEGRATION

Eric M. Vestrup

WILEY-INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

This text is printed on acid-free paper. ♾

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication is available.

Vestrup, Eric M., 1971–
The theory of measures and integration / Eric M. Vestrup.
p. cm. — (Wiley series in probability and statistics)
Includes bibliographical references and index.
ISBN 0-471-24977-7 (acid-free paper)
1. Measure theory. 2. Integrals, Generalized. I. Title. II. Series.
QA325.V47 2004
515'.42—dc21

2003053475

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

To my parents,
Raymond R. Vestrup
and
Linda M. Vestrup

This Page Intentionally Left Blank

Contents

This Page Intentionally Left Blank

Preface

The goal of this book is to present the major results of classical measure and integration theory in as clear, rigorous, and detailed a fashion as possible. This book assumes that the reader has studied advanced calculus and elementary analysis. For example, the first eight chapters of Walter Rudin's *Principles of Mathematical Analysis* should provide a strong preparatory framework for the material in this book.

Measure Theory and its sister Integration Theory are viewed by some as special topics to be subsumed under the larger heading of *Analysis*, and many analysis books cover these topics in varying degrees of detail. However, these topics are rich enough that they can stand alone as deep and fascinating topics. Modern probability theory and mathematical statistics rely heavily on measure and integration theory, and to understand the former topics, one must be competent in the latter topics.

In the writing of proofs and in my choice of notation, I have tried to be as explicit as possible. Elegance or a sense of aesthetics have without exception deferred to clarity, although one can certainly be both clear and elegant in certain situations. If anything, I have erred on the side of perhaps showing too many details, although I believe most readers will at some time be grateful for the details.

Anybody who knows the area well and purviews this book will see the influence of the texts of Walter Rudin, Kai Lai Chung, and Paul Halmos. One will also see, in tone and in the style of presentation, the influence of those magisterial and stupendous works *Real and Abstract Analysis* by Edwin Hewitt and Karl Stromberg, and *Probability and Measure* by Patrick Billingsley. All of these men and their works were my teachers in this area, and any apparent profundity or excellence on my part is merely a reflection of their mastery and presentation. I make absolutely no claim to originality in this book, as all of the material within is quite classical. My contribution comes [so I hope] in making certain difficult areas of the classical material very clear and accessible to the reader.

There are exercises found at the end of almost every section. These exercises range from being trivialities to being quite substantial developments in and of themselves. Hints or outlines are provided for many of the more substantive claims made in the exercises. As Hewitt and Stromberg say in their introduction, the more heroic readers may ignore the hints and outlines, but the hope is that every reader will be grateful for some of them. Certain exercises are marked with an asterisk; these exercises are results that are used implicitly or explicitly in later work. I believe that some of the exercises, besides providing useful extensions or side comments pertaining to the results in the text, are also interesting on their own merits. Exercises are quite often stated as assertions. For example, if *P has property X* appears in the exercise, this should be interpreted as *Give a proof of the claim that P has property X*. At any rate, the reader shouldn't have any problem figuring out exactly what is asked of him in the exercises. As with the material in the text, the exercises are again of a classical nature; I again make no claim of originality in this regard.

I believe that this book could serve as a primary or supplementary text for (1) a semester-long or year-long real analysis sequence that deals heavily with integration and measure theory, (2) a complementary background text for graduate courses in analysis, probability, and mathematical statistics, and, of course, (3) self study by those with mathematical maturity and the requisite background. This book is solidly "graduate-level" if we average over all of the contents. [There are some areas that solid and uncommonly mature undergraduates would find accessible, and there are some areas that require set theoretic considerations that are usually not found until later in graduate school, if it all.]

It may be helpful to briefly discuss the material in this book, so a relatively short description of the chapters will now be given.

Chapter 1. Sections 1-5 are required, and Sections 6 and 7 are optional. The first five sections are straightforward, and shouldn't pose much of a problem. I have tried to give the reader as explicit a description of Borel sets as possible without being too pedantic. Sections 6 and 7 are fairly deep and involved, but the advanced reader may find them worth the study. With only a few very tangential exceptions, nothing from Sections 6 and 7 will be used in later work.

Chapter 2. Those who merely want to study measures in the abstract can possibly get by with only the first two sections. However, this would require missing out on the construction of Lebesgue measure, which surely is an important item. Therefore, Sections 3 and 4 are also highly recommended. The first two sections are used everywhere in the text. Sections 3 and 4 are used primarily in Chapters 3, 4, and 7 when dealing with Euclidean space. The instructor should at least cover Sections 1-3 thoroughly.

Chapter 3. This is where we obtain our first collection of deep and foundational results. A minimalist approach covers the first six sections, although really all ten sections should be covered in detail. This chapter is far deeper than the first two chapters, representing a collection of some of the most important results in measure theory.

Chapter 4. Sections 1-4 are required, as they discuss our rigorous formulation of length, area, volume, and so on. These sections serve as a sort of apologetic for all of the work done in Chapters 2 and 3 in the sense that we show that Lebesgue measure does everything [or almost everything] that we expect length, area, volume, [and so on] to do. Sections 5-9 are optional, but in my opinion they are fascinating on their own merits. It is my hope that those who read these latter sections find the results of interest.

Chapter 5. Sections 1-7 are required, while Section 8 is optional. Like Chapters 1 and 2, this section is really nothing more than a collection of definitions and some basic results that may not be seen as anything more than necessary evils for the more interesting things to come. This chapter in essence is the grab-bag of the results needed to construct the Lebesgue integral in Chapter 6.

Chapter 6. This chapter is the very soul of the book, and nothing can be safely skipped by the reader. In this chapter, we painstakingly construct the abstract Lebesgue integral, proving many properties concerning it. This chapter will be heavily used in all that follows.

Chapter 7. This fairly short chapter demonstrates that Lebesgue and Riemann integrals coincide in a large class of situations. Those pressed for time may skip this chapter and take its results for granted. However, there is nothing particularly hard about the theory. Section 5 may seem unimportant at first, but it will be used in the [optional] discussion of convolutions in Chapter 10. I would have liked to include a careful buildup to a general change of variables result for multiple integrals, but space considerations prevent such a buildup. The same space considerations prevent a detailed discussion of the famous theorems of Lebesgue that relate derivatives to Lebesgue integrals.

Chapter 8. This and the following three chapters present the main avenues through which one takes the theory of Chapter 6. For this sizable chapter, the first seven sections are absolutely required, as these results and discussions are foundational for both analysis and measure theory. Sections 8 and 9 may be thought of as optional, especially in light of the fact that the main result of these two sections will be obtained [in a slightly weaker form] using the techniques of Chapter 9.

Chapter 9. Section 1 is essential, presenting the famous and important Radon-Nikodym Theorem. Section 2 is a fairly deep section, and some may justifiably find that the generalization to the Radon-Nikodym Theorem obtained in that section is not worth all of the extra work. It is there for those who are interested, and those who are not interested [or are pressed for time] need not worry that this section will be needed for later work. Section 3 presents a second proof of the Riesz Representation Theorem as originally given in Section 8.9. Readers who skipped Sections 8.8 and 8.9 might want to study this section so that the Riesz Representation Theorem may be seen at least once. Section 4 is a long and involved section that may be safely skipped by those who are not planning to study Chapter 11, or who want to study the Strong Laws in Chapter 11 without spending so much time on the constructions given there. [Section 4 in no way requires Sections 2 or 3.] For those who will want all of the details behind the Strong Laws, this section is absolutely essential, forming a key component for the convergence results given in Section 3 of Chapter 11.

Chapter 10. This is the longest chapter in the book, and presents the basic theory regarding product measures and integrals over product spaces. Despite the length of this chapter, those who are in a hurry can get by with merely covering the first two sections, where a very complete and lucid presentation of the standard theory is given. The remaining four sections present interesting applications of the Fubini Theorems and the general theory of product measures, and are there for those who want to see "Fubini in action" and the like. However, these four sections are not used in Chapter 11, so the lecturer may pretty much do as he pleases with these sections. It is hoped that some readers will appreciate the discussion of convolutions and the famous Hardy-Littlewood Maximal Theorem.

Chapter 11. Chapter 11 discusses the product of two measure spaces, while this chapter discusses arbitrary products of measure spaces. There are five sections in this chapter. The first two sections carefully [some might say tediously] construct the requisite items, showing that certain objects actually exist and make sense. Depending on one's tastes and desires, the work produced by these two foundational sections may be viewed as interesting or an annoyance. Those who are under a time constraint might be able to get away with covering the notation and definitions, while presenting the basic ideas and results without spending a lengthy amount of time on their proofs. However, part of measure theory and mathematics in general is a quest for rigor and details, so I would encourage everybody to study the first two sections, painful as they might be. Section 3 of this chapter may be thought of as a first application of the theory of Sections 1 and 2, or it may be regarded as more machinery needed to state and prove the Strong Laws of Large Numbers. The same comments regarding Sections 1 and 2 apply to Section 3. Sections 4–6 represent the pinnacle of the chapter [and perhaps of the entire book], where careful proofs of the famous Strong Laws are presented. It would have been of interest to present some further topics regarding infinite-dimensional product spaces, such as Martingale Theory, Kakutani's Dichotomy, Brownian Motion, and a general treatment of stochastic processes, but limitations of space [and the author's energy!] and a general desire to not wander too far from the beaten path prevent their inclusion.

My hope has been to write well enough and clearly enough that even the most advanced concepts appear accessible to general readers pursuing a Ph.D. or graduate degree in mathematics, statistics, or a field that requires a rigorous background in integration and measure theory. Those who read the book will of course be able to make up their own minds on whether I have succeeded in this regard.

Eric Vestrup

Downers Grove
Illinois

This Page Intentionally Left Blank

Acknowledgments

Many people have directly or indirectly contributed to this book. I would like to acknowledge the great amounts of help given by Professors Alan Fenech, Wesley Johnson, Prabir Burman, Ed Mack, Jessica Utts, Rahman Azari, Robert Shumway, Rich Levine, P. K. Bhattacharya, and George Roussas in matters mathematical and nonmathematical during my times as a graduate student and then as a professor. Special mention must be made of Professor Francisco Samaniego, under whom I studied and wrote my dissertation, and with whom I still have the pleasure of working.

Outside of UC Davis, I would next like to thank Professors Jonathan Cohen, Jeffrey Bergen, Jerry Goldman, Roger Jones, Stefan Catoiu, Constantine Georgakis, Gang Wang, Ayse Sahin, Effat Moussa, and David Webb for helpful assistance in mathematical matters and such.

Also, the following professors at Sacramento State University [my undergraduate school] are acknowledged: Geetha and Doraiswamy Ramachandran, Tom Schulte, Earl Kymala, and Don Taranto.

I next wish to thank Cathy Hertz, whose editing saved many errors from seeing the light of day.

Finally, the acknowledgement that is supreme must be made: I thank the Triune God of the Old and New Testaments for allowing me to finish this project in one piece and on time.

1

Set Systems

This chapter talks about the various collections of sets associated with measure theory. The need for these many set structures defined and discussed within comes from the desire to rigorously define the framework wherein measure-theoretic concepts reside. Of the seven sections in this chapter, the first five are essential. The last two sections are optional and show the profundity of certain seemingly simple concepts.

Preliminary Notation. We will use the standard symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{R}$ to denote the sets of positive integers, rational numbers, integers, and real numbers respectively. Note that $\pm\infty \notin \mathbb{R}$. There are 9 types of intervals in $\mathbb{R}$, and we use standard notation for them:

$$(-\infty,\infty),\ (-\infty,x],\ (-\infty,x),\ [x,\infty),\ (x,\infty),\ (a,b),\ (a,b],\ [a,b),\ [a,b],$$

where $x, a, b \in \mathbb{R}$. In particular, intervals of the form $(a, b]$ are called *right semiclosed* [*rsc*]; intervals of the form $[a, b)$ are called *left semiclosed intervals* [*lsc*].

For $k \in \mathbb{N}$, the set $\mathbb{R}^k$ is called *k-dimensional Euclidean space,* and $\mathbb{R}^k$ is defined as the k-fold Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$. Given $\boldsymbol{x} \in \mathbb{R}^k$, we will write $\boldsymbol{x} = (x_1, \cdots, x_k)$, where $x_1, \cdots, x_k \in \mathbb{R}$. The nine types of intervals above have their k-dimensional analogs, denoted by

$$(-\boldsymbol{\infty},\boldsymbol{\infty}),\ (-\boldsymbol{\infty},\boldsymbol{x}],\ (-\boldsymbol{\infty},\boldsymbol{x}),\ [\boldsymbol{x},\boldsymbol{\infty}),\ (\boldsymbol{x},\boldsymbol{\infty}),\ (\boldsymbol{a},\boldsymbol{b}),\ (\boldsymbol{a},\boldsymbol{b}],\ [\boldsymbol{a},\boldsymbol{b}),\ [\boldsymbol{a},\boldsymbol{b}],$$

where $\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$. The interpretations of these k-dimensional rectangles is similar to the one-dimensional case: $(-\boldsymbol{\infty}, \boldsymbol{x}]$ denotes the set of $\boldsymbol{y} \in \mathbb{R}^k$

with $-\infty < y_i \le x_i$ for $i = 1, \cdots, k$; $(\boldsymbol{a}, \boldsymbol{b}]$ denotes the set of $\boldsymbol{x} \in \mathbb{R}^k$ with $a_i < x_i \le b_i$ for $i = 1, \cdots, k$, and so on. We will write $\boldsymbol{x} \le \boldsymbol{y}$ [or $\boldsymbol{x} < \boldsymbol{y}$] to mean that $x_i \le y_i$ [or $x_i < y_i$] for $i = 1, \cdots, k$. Thus, we may write $[\boldsymbol{x}, \infty) = \{\boldsymbol{y} \in \mathbb{R}^k : \boldsymbol{y} \ge \boldsymbol{x}\}$, $(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^k : \boldsymbol{a} < \boldsymbol{x} < \boldsymbol{b}\}$, etc. We will refer to $(\boldsymbol{a}, \boldsymbol{b}]$ as a *rsc rectangle*; $[\boldsymbol{a}, \boldsymbol{b})$ is a *lsc rectangle*.

The set $\bar{\mathbb{R}}$ of *extended real numbers* is defined as $\mathbb{R} \cup \{-\infty, +\infty\}$. While $\pm\infty \notin \mathbb{R}$, we do have $\pm\infty \in \bar{\mathbb{R}}$. By definition, $-\infty < +\infty$ and $-\infty < x < +\infty$ for all $x \in \mathbb{R}$. Subsets of $\bar{\mathbb{R}}$ having the form A, $A \cup (x, \infty]$, $A \cup [-\infty, x)$, or $A \cup [-\infty, x) \cup (y, \infty]$, where $A \subseteq \mathbb{R}$ is open and $x, y \in \mathbb{R}$, are declared open. In particular, both $\varnothing$ and $\bar{\mathbb{R}}$ are open subsets of $\bar{\mathbb{R}}$.

For $k \in \mathbb{N}$, the set $\bar{\mathbb{R}}^k$ is called *k-dimensional extended Euclidean space*, and is the k-fold Cartesian product of $\bar{\mathbb{R}}$. Therefore, an element $\boldsymbol{x} \in \bar{\mathbb{R}}^k$ has some, none, or possibly all of its coordinates equal to $+\infty$ and/or $-\infty$, and all other coordinates are real numbers. We write $+\boldsymbol{\infty} = (+\infty, \cdots, +\infty)$ and $-\boldsymbol{\infty} = (-\infty, \cdots, -\infty)$.

Given a set Ω, we denote its cardinal number by card(Ω). We will write $\mathfrak{c} = \text{card}(\mathbb{R})$, $\aleph_0 = \text{card}(\mathbb{N})$, and $\mathfrak{2} = \text{card}(\{0, 1\})$. The set of all subsets of Ω [or the *power set of* Ω] is denoted by 2^Ω; we write $\mathfrak{2}^{\text{card}(\Omega)}$ for card(2^Ω). A set Ω is *at most countable* [amc] iff Ω is finite or $\text{card}(\Omega) = \aleph_0$, that is, if $\text{card}(\Omega) \le \aleph_0$. Ω will be called *uncountable* iff it is not amc.

1.1 π-SYSTEMS, λ-SYSTEMS, AND SEMIRINGS

This section deals with three systems of sets used in measure theory, and is broken up into three subsections, each with exercises.

Definition. A nonvoid collection $\mathcal{P}$ of subsets of a nonvoid set Ω is called a *π-system [of subsets of Ω]* iff $A, B \in \mathcal{P}$ implies $A \cap B \in \mathcal{P}$, that is, $\mathcal{P}$ is *closed under intersection.* We will also say that $\mathcal{P}$ *is a π-system on/over* Ω, or, if no confusion can arise, $\mathcal{P}$ *is a π-system.*

Example 1. Let (Ω, ρ) denote a metric space. Recall that if $A, B \subseteq \Omega$ and both A and B are open with respect to ρ, then $A \cap B$ is open. Thus, the collection of open subsets of Ω forms a π-system of subsets of Ω. Next, recall that the intersection of two closed subsets of Ω is itself a closed subset of Ω, and hence the collection of closed subsets of Ω is a π-system as well.

Example 2. Let $\Omega = \mathbb{R}$, and let $\mathcal{P}$ consist of $\varnothing$ and the rsc intervals. It may be painlessly verified that $\mathcal{P}$ is a π-system on $\mathbb{R}$. Next, let $\Omega = \mathbb{R}^k$, and let $\mathcal{P}_k$ consist of $\varnothing$ and the k-dimensional rsc rectangles. *We claim that $\mathcal{P}_k$ is a π-system.* To see this, let $A, B \in \mathcal{P}_k$. If A or B is empty, then $A \cap B = \varnothing \in \mathcal{P}_k$. If $A = (\boldsymbol{a}, \boldsymbol{b}]$ and $B = (\boldsymbol{c}, \boldsymbol{d}]$, then $A \cap B = C_1 \times \cdots \times C_k$, where $C_i = (a_i, b_i] \cap (c_i, d_i]$, $i = 1, \cdots k$. Since $\mathcal{P}$ is a π-system, we have $C_1, \cdots, C_k \in \mathcal{P}$ by what has been shown in the case $k = 1$. Since a k-fold Cartesian product of $\mathcal{P}$-sets is a $\mathcal{P}_k$-set, we have $A \cap B \in \mathcal{P}_k$, and thus $\mathcal{P}_k$ is closed under intersection.

In general, it is automatic that a nonempty collection $\mathcal{P}$ is a π-system on a nonempty set Ω iff for every finite collection $A_1, \cdots, A_n$ of $\mathcal{P}$-sets, we have $\bigcap_{i=1}^n A_i \in \mathcal{P}$ *[closure under finite intersections]*. At least two π-systems of subsets of Ω always exist: 2^Ω and the collection $\{\varnothing, \Omega\}$. Thus, the concept of a π-system is never logically vacuous.

Exercises.

1*. Let $\Omega = (\alpha, \beta]$. Let $\mathcal{P}$ consist of $\varnothing$ along with the rsc subintervals of Ω. $\mathcal{P}$ is a π-system of subsets of $(\alpha, \beta]$.

2. Must $\varnothing$ be in every π-system?

3. List all π-systems consisting of at least two subsets of $\{\omega_1, \omega_2, \omega_3\}$.

4*. If $\mathcal{P}$ consists of the empty set and the k-dimensional rectangles of any one form, then $\mathcal{P}$ is a π-system of subsets of $\mathbb{R}^k$.

5. Let $\mathcal{P}$ consist of $\varnothing$ and all subsets of $\mathbb{R}^k$ that are neither open nor closed. Then $\mathcal{P}$ *is not a* π*-system of subsets of* $\mathbb{R}^k$.

6*. For each α in a nonempty index set A, let $\mathcal{P}_\alpha$ be a π-system over Ω.
(a) The collection $\bigcap_{\alpha \in A} \mathcal{P}_\alpha$ is a π-system on Ω.
(b) Let $\mathcal{A} \subseteq 2^\Omega$. Suppose that $\{\mathcal{P}_\alpha : \alpha \in A\}$ is the "exhaustive list" of all the π-systems that contain $\mathcal{A}$. In other words, each $\mathcal{P}_\alpha \supseteq \mathcal{A}$, and any π-system that contains $\mathcal{A}$ coincides with some $\mathcal{P}_\alpha$. Then $\bigcap_{\alpha \in A} \mathcal{P}_\alpha$ is a π-system that contains $\mathcal{A}$. If $\mathcal{Q}$ is a π-system containing $\mathcal{A}$, then $\bigcap_{\alpha \in A} \mathcal{P}_\alpha \subseteq \mathcal{Q}$. [The collection $\bigcap_{\alpha \in A} \mathcal{P}_\alpha$ is called the *[minimal]* π*-system [on* Ω*] generated by* $\mathcal{A}$.] The minimal π-system generated by $\mathcal{A}$ always exists.
(c) Suppose that $\mathcal{P}$ is a π-system with $\mathcal{P} \supseteq \mathcal{A}$, and suppose that $\mathcal{P}$ is contained in any other π-system that contains $\mathcal{A}$. Then $\mathcal{P} = \bigcap_{\alpha \in A} \mathcal{P}_\alpha$, with notation as in (b). The minimal π-system containing $\mathcal{A}$ [which always exists] is also unique.

It will turn out that π-systems play a role in an important uniqueness question in Chapter 3. At this stage, with the definition and the exercises, there is not much else to say. The next definition will give us a collection of sets that has a stronger set of closure properties.

Definition. A nonempty collection $\mathcal{L}$ of subsets of a nonempty set Ω is called a λ*-system [on* Ω*]* iff
(λ_1) $\Omega \in \mathcal{L}$,
(λ_2) $A \in \mathcal{L}$ implies $A^c \in \mathcal{L}$, and
(λ_3) For every disjoint sequence $\{A_n\}_{n=1}^\infty$ of $\mathcal{L}$-sets, we have $\bigcup_{n=1}^\infty A_n \in \mathcal{L}$.

Properties (λ_2) and (λ_3) are called *closure under complementation* and *closure under countable disjoint unions*, respectively. By (λ_1) and (λ_2), it is automatic that $\varnothing$ is in every λ-system. Next, observe that the collections $\{\varnothing, \Omega\}$ and 2^Ω are λ-systems, hence a λ-system over Ω always exists.

Example 3. For any $k \in \mathbb{N}$, if Ω consists of $2k$ elements, then the collection that consists of Ω, $\varnothing$, and all k-element subsets of Ω is a λ-system of subsets of Ω.

The additional closure properties listed below will be used in the exercises:

(λ_2') For all $A, B \in \mathcal{L}$, $A \subseteq B$ implies $B - A \in \mathcal{L}$;
(λ_4) For all $A, B \in \mathcal{L}$, $A \cap B = \varnothing$ implies $A \cup B \in \mathcal{L}$;
(λ_5) For any nondecreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{L}$-sets, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$;
(λ_6) For any nonincreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{L}$-sets, $\bigcap_{n=1}^{\infty} A_n \in \mathcal{L}$.

These properties are called *closure under proper differences*, *closure under disjoint unions*, *closure under nondecreasing countable unions*, and *closure under nonincreasing countable intersections*. We use the term *nondecreasing* relative to $\{A_n\}_{n=1}^{\infty}$ to mean that $A_1 \subseteq A_2 \subseteq \cdots$, where any of the containment relations may in fact be equality. We say that $A_1, A_2, \cdots$ are *[strictly] increasing* if each of the containments is proper: $A_1 \subsetneq A_2 \subsetneq \cdots$. Similar comments apply to the terms *nonincreasing* $[A_1 \supseteq A_2 \supseteq \cdots]$ and *[strictly] decreasing* $[A_1 \supsetneq A_2 \supsetneq \cdots]$.

Exercises.

7*. This exercise explores some equivalent definitions of a λ-system.
(a) $\mathcal{L}$ is a λ-system iff $\mathcal{L}$ satisfies (λ_1), (λ_2'), and (λ_3).
(b) Every λ-system additionally satisfies (λ_4), (λ_5), and (λ_6).
(c) $\mathcal{L}$ is a λ-system iff $\mathcal{L}$ satisfies (λ_1), (λ_2'), and (λ_5).
(d) If a collection $\mathcal{L}$ is nonempty and satisfies (λ_2) and (λ_3), then $\mathcal{L}$ is a λ-system.

8*. If $\mathcal{L}$ is a λ-system and a π-system, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$ whenever $A_n \in \mathcal{L}$ for all $n \in \mathbb{N}$. That is, $\mathcal{L}$ is *closed under countable unions*.

9. A λ-system is not necessarily a π-system.

10. Find all λ-systems over $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ with at least three elements.

11. The collection consisting of $\varnothing$ and the rsc intervals is not a λ-system on $\mathbb{R}$.

12*. Suppose that for each α in a nonempty index set A, $\mathcal{L}_\alpha$ is a λ-system over Ω.
(a) The collection $\bigcap_{\alpha \in A} \mathcal{L}_\alpha$ is a λ-system on Ω.
(b) Suppose that $\mathcal{A} \subseteq 2^\Omega$ is such that $\mathcal{A}$ is contained in each $\mathcal{L}_\alpha$, and suppose that $\{\mathcal{L}_\alpha : \alpha \in A\}$ is the "exhaustive list" of all the λ-systems that contain $\mathcal{A}$. Then $\bigcap_{\alpha \in A} \mathcal{L}_\alpha$ is a λ-system that contains $\mathcal{A}$. If $\mathcal{Q}$ is a λ-system on Ω that contains $\mathcal{A}$, then $\bigcap_{\alpha \in A} \mathcal{L}_\alpha \subseteq \mathcal{Q}$. [The collection $\bigcap_{\alpha \in A} \mathcal{L}_\alpha$ is also called the *[minimal] λ-system [over Ω] generated by* $\mathcal{A}$.] The minimal λ-system generated by $\mathcal{A}$ always exists.
(c) Let $\mathcal{L}$ denote a λ-system over Ω with $\mathcal{L} \supseteq \mathcal{A}$ and where $\mathcal{L}$ is contained in any other λ-system also containing $\mathcal{A}$. Then $\mathcal{L} = \bigcap_{\alpha \in A} \mathcal{L}_\alpha$, with notation as in (b). Therefore, the λ-system generated by $\mathcal{A}$ always exists and is unique.

The third type of collection of sets in this section will now be presented.

Definition. A *semiring on/over* Ω is a collection $\mathcal{A}$ of subsets of Ω satisfying
(SR1) $\varnothing \in \mathcal{A}$,
(SR2) $\mathcal{A}$ is a π-system, and
(SR3) If $A, B \in \mathcal{A}$ with $A \subseteq B$, then there exist disjoint $C_1, \cdots, C_k \in \mathcal{A}$ with $B - A = C_1 \cup \cdots \cup C_k$. [Equivalently, we have $B = A \cup C_1 \cup \cdots \cup C_k$.]

Example 4. Both 2^Ω and $\{\varnothing, \Omega\}$ are [trivial] semirings of subsets of Ω.

Example 5. Let $\Omega = \mathbb{R}$, and let $\mathcal{A}$ consist of $\varnothing$ and all rsc intervals. Property (SR1) is automatic, and Example 2 gives (SR2). To verify (SR3), pick $A \subseteq B$

with $A, B \in \mathcal{A}$. If $A = \varnothing$, then $B - A = B$ and (SR3) trivially holds. Next, let $A = (a, b]$ and $B = (c, d]$ with $c \leq a < b \leq d$, and consider the cases (i) $a = c$ and $b = d$, (ii) $a = c$ and $b < d$, (iii) $c < a$ and $b = d$, and (iv) $c < a$ and $b < d$. For these cases, $B - A$ is $\varnothing$, $(b, d]$, $(c, a]$, and $(c, a] \cup (b, d]$, respectively. In every case (SR3) holds, hence $\mathcal{A}$ *is a semiring on* $\mathbb{R}$. *If we instead used* lsc *intervals,* $\mathcal{A}$ would still be a semiring, but if we used intervals of any other form $\mathcal{A}$ would fail to be a semiring. This consideration is why we will concentrate on rsc intervals when working in subsequent chapters.

Example 6. Let $\Omega = \mathbb{R}^k$, and let $\mathcal{A}_k$ consist of $\varnothing$ and the k-dimensional rsc rectangles. *We claim that* $\mathcal{A}_k$ *is a semiring.* (SR1) is satisfied by definition, and Example 2 shows that (SR2) holds. We now turn to verifying (SR3). Let $A, B \in \mathcal{A}_k$ with $A \subseteq B$. If $A = \varnothing$, then $B - A = B$, and (SR3) trivially holds in this case. Otherwise, suppose that $A \neq \varnothing$ and write $A = I_1 \times \cdots \times I_k$ and $B = J_1 \times \cdots \times J_k$, where each of $I_1, \cdots, I_k, J_1, \cdots, J_k$ is a rsc interval. Since $A \subseteq B$, we have $I_i \subseteq J_i$, $i = 1, \cdots, k$. By Example 5, $J_i - I_i$ is a disjoint union $A_i \cup B_i$, where A_i and B_i freely denote rsc intervals or $\varnothing$, $i = 1, \cdots, k$. Consider the 3^k k-dimensional sets $C_1 \times \cdots \times C_k$, where C_i denotes either I_i, A_i, or B_i, $i = 1, \cdots k$; the sets $C_1 \times \cdots \times C_k$ are disjoint [some might be empty]. One of the sets $C_1 \times \cdots \times C_k$ is A, and $B - A$ is the disjoint union of the remaining sets $C_1 \times \cdots \times C_k$. Since each set of the form $C_1 \times \cdots \times C_k$ is either empty or a k-dimensional rsc rectangle, (SR3) holds.

The role of semirings will become apparent in discussing the results relating to the extension and uniqueness of measures, found in Chapter 3.

Exercises.

13*. Is $\{\varnothing\} \cup \{(0, x] : 0 < x \leq 1\}$ a semiring over $(0, 1]$?

14*. This exercise explores some alternative definitions of a semiring.
(a) Some define $\mathcal{A}$ to be a semiring iff $\mathcal{A}$ is a nonempty π-system such that both $E, F \in \mathcal{A}$ and $E \subseteq F$ imply the existence of a finite collection $C_0, C_1, \cdots, C_n \in \mathcal{A}$ with $E = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq F$ and $C_i - C_{i-1} \in \mathcal{A}$ for $i = 1, \cdots, n$. This definition of a semiring is equivalent to our definition of a semiring.
(b) Some define $\mathcal{A}$ to be a semiring by stipulating (SR1), (SR2), and the following property: $A, B \in \mathcal{A}$ implies the existence of disjoint $\mathcal{A}$-sets $C_0, C_1, \cdots, C_n$ with $B - A = \bigcup_{i=0}^{n} C_i$. Note that here $B - A$ is not necessarily a *proper* difference. If $\mathcal{A}$ is a semiring by this definition, then $\mathcal{A}$ is a semiring by our definition, but the converse is not necessarily true.

15*. Let $\mathcal{A}$ consist of $\varnothing$ as well as all rsc rectangles $(\boldsymbol{a}, \boldsymbol{b}]$. The collection of all finite disjoint unions of $\mathcal{A}$-sets is a semiring over $\mathbb{R}^k$.

16. An arbitrary intersection of semirings on Ω is not necessarily a semiring on Ω.

17. If $\mathcal{A}$ is a semiring over Ω, must $\Omega \in \mathcal{A}$?

18*. Let $\mathcal{A}$ denote a semiring. Pick $n \in \mathbb{N}$, and let $A, A_1, \cdots, A_n \in \mathcal{A}$. Then there exists a finite collection $\{C_1, \cdots, C_m\}$ of disjoint $\mathcal{A}$-sets with $A - \bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} C_j$. [When $n = 1$, write $A - A_1 = A - (A \cap A_1)$ and invoke (SR3). Assume the result is true for a fixed n and show that $A - \bigcup_{i=1}^{n+1} A_i = \bigcup_{j=1}^{m} (C_j - A_{n+1})$. Apply the $n = 1$ case to each of the $C_j - A_{n+1}$ terms.]

19. Other books deal with a system called a *ring*. We will not deal with rings of sets in this text, but since the reader might refer to other books that deal with rings, it is worthy to discuss the concept. A collection $\mathcal{R}$ of subsets of a nonempty set Ω is called a *ring* of subsets of Ω iff (R1) $\mathcal{R} \neq \varnothing$, (R2) $A, B \in \mathcal{R}$ implies $A \cup B \in \mathcal{R}$, and (R3) $A, B \in \mathcal{R}$ implies $A - B \in \mathcal{R}$. That is, a ring is a nonempty collection of subsets closed under unions and differences.
(a) $\varnothing$ is in every ring.
(b) $\mathcal{R}$ is a ring iff $\mathcal{R}$ satisfies (R1), (R2), and (R4): $A, B \in \mathcal{R}$ with $A \subseteq B$ implies $B - A \in \mathcal{R}$. [Use the identity $B - A = (B \cup A) - A$.]
(c) Every ring satisfies (R5): $A, B \in \mathcal{R}$ implies $A \Delta B \in \mathcal{R}$. [$A \Delta B$ is the usual *symmetric difference of A and B*, the set of elements that are in exactly one of A or B. That is, $A \Delta B = (A - B) \cup (B - A)$.]
(d) Every ring is a π-system. [Verify first that $A \cap B = (A \cup B) - (A \Delta B)$.]
(e) Every ring is closed under finite unions and finite intersections.
(f) $\mathcal{R}$ is a ring iff $\mathcal{R}$ is a nonempty π-system that satisfies (R4) along with (R6): $A, B \in \mathcal{R}$ and $A \cap B = \varnothing$ imply $A \cup B \in \mathcal{R}$. [First, show that $A \cup B$ coincides with $[A - (A \cap B)] \cup [B - (A \cap B)] \cup (A \cap B)$.]
(g) $\mathcal{R}$ is a ring iff $\mathcal{R}$ is a nonempty π-system that satisfies (R5). [First, verify the identities $A \cup B = (A \Delta B) \Delta (A \cap B)$ and $A - B = A \Delta (A \cap B)$.]
(h) Suppose that $\{\mathcal{R}_\alpha : \alpha \in A\}$ is the "exhaustive list" of all rings that contain $\mathcal{A}$. Then $\bigcap_{\alpha \in A} \mathcal{R}_\alpha$ is a ring that contains $\mathcal{A}$, and $\bigcap_{\alpha \in A} \mathcal{R}_\alpha$ is contained in any ring that contains $\mathcal{A}$. [The collection $\bigcap_{\alpha \in A} \mathcal{R}_\alpha$ is called the *[minimal] ring generated by* $\mathcal{A}$.] The minimal ring containing $\mathcal{A}$ always exists and is unique.
(i) The collection of finite unions of rsc intervals is a ring on $\mathbb{R}$.
(j) Let Ω be uncountable. The collection of all amc subsets of Ω is a ring on Ω.

20. This problem explores the relationship between semirings and rings.
(a) Every ring is a semiring. However, not every semiring is a ring.
(b) Let $\mathcal{A}$ denote a semiring on Ω, and let $\mathcal{R}$ consist of the finite disjoint unions of $\mathcal{A}$-sets. Then $\mathcal{R}$ is closed under finite intersections and disjoint unions.
(c) If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $B - A \in \mathcal{R}$.
(d) $A \in \mathcal{A}$, $B \in \mathcal{R}$, and $A \subseteq B$ imply $B - A \in \mathcal{R}$.
(e) $A, B \in \mathcal{R}$ and $A \subseteq B$ imply $B - A \in \mathcal{R}$.
(f) $\mathcal{R}$ is the minimal ring generated by $\mathcal{A}$. [See Exercise 19(h).]
(g) A semiring that satisfies (R2) is a ring.

21. Let Ω be infinite, and let $\mathcal{A} \subseteq 2^\Omega$ have cardinality $\aleph_0$. We will show that the ring generated by $\mathcal{A}$ has cardinality $\aleph_0$.
(a) Given $\mathcal{C} \subseteq 2^\Omega$, let $\mathcal{C}^*$ denote the collection of all finite unions of differences of $\mathcal{C}$-sets. If $\text{card}(\mathcal{C}) = \aleph_0$, then $\text{card}(\mathcal{C}^*) = \aleph_0$. Also, $\varnothing \in \mathcal{C}$ implies $\mathcal{C} \subseteq \mathcal{C}^*$.
(b) Let $\mathcal{A}_0 = \mathcal{A}$, and define $\mathcal{A}_n = \mathcal{A}^*_{n-1}$ for $n \geq 1$. Then $\mathcal{A} \subseteq \bigcup_{n=0}^\infty \mathcal{A}_n \subseteq R(\mathcal{A})$, where $R(\mathcal{A})$ is the minimal ring generated by $\mathcal{A}$ and where [without loss of generality] $\varnothing \in \mathcal{A}$. Also, $\text{card}(\bigcup_{n=0}^\infty \mathcal{A}_n) = \aleph_0$.
(c) $\bigcup_{n=0}^\infty \mathcal{A}_n$ is a ring on Ω, and from the fact that $R(\mathcal{A})$ is the minimal ring containing $\mathcal{A}$, we have $\bigcup_{n=0}^\infty \mathcal{A}_n = R(\mathcal{A})$, and thus $\text{card}(R(\mathcal{A})) = \aleph_0$.
(d) We may generalize: if $\mathcal{A}$ is infinite, then $\text{card}(\mathcal{A}) = \text{card}(R(\mathcal{A}))$.

1.2 FIELDS

This section deals with a new system of sets that is more stringent in its requirements than are the systems of the previous section. We will call this new system a *field* of sets. As before, Ω denotes a nonempty set.

Definition. A *field on* Ω is a collection $\mathcal{F} \subseteq 2^{\Omega}$ such that
(F1) $\Omega \in \mathcal{F}$,
(F2) $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$, and
(F3) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

Using induction, it is easy to see that every field is closed under arbitrary finite unions and intersections: if $A_1, \cdots, A_n \in \mathcal{F}$ then $\bigcup_{i=1}^n A_i$, $\bigcap_{i=1}^n A_i \in \mathcal{F}$. It is also easily verified that every field is a π-system and a semiring; however, a field is not necessarily a λ-system. Also, $\mathcal{F}$ is a field iff $\mathcal{F}$ satisfies (F1), (F2), and is a π-system; this latter characterization is often useful.

Example 1. Both 2^{Ω} and $\{\varnothing, \Omega\}$ are trivial fields of subsets of Ω.

Example 2. Let $\Omega = (0,1]$ and let $\mathcal{F}$ consist of $\varnothing$ and all rsc subintervals $(a,b]$ of $(0,1]$. Note immediately that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is a π-system, but requirement (F2) might fail: any $A = (a,b]$ with $0 < a < b < 1$ will be such that A^c is neither $\varnothing$ nor a rsc subinterval, and hence $A^c \notin \mathcal{F}$. Accordingly, $\mathcal{F}$ is not a field over Ω.

Example 3. Keep $\Omega = (0,1]$ as in the previous example, but amend $\mathcal{F}$ to consist of $\varnothing$ and the *finite disjoint unions* of rsc subintervals of $(0,1]$. That is, a non-empty $A \in \mathcal{F}$ iff for some $n \in \mathbb{N}$ we have $A = \bigcup_{i=1}^n (a_i, b_i]$, where $(a_i, b_i] \subseteq (0,1]$ for $i = 1, \cdots, n$ and the union is disjoint. Note that this definition of $\mathcal{F}$ includes the sets in the collection from the above example.

The verification that $\mathcal{F}$ is a field is a direct one. Clearly $\Omega \in \mathcal{F}$, so (F1) holds. To verify (F2), pick $A \in \mathcal{F}$. If $A = \varnothing$, then $A^c = (0,1] = \Omega \in \mathcal{F}$. If $A = (0,1] = \Omega$, then $A^c = \varnothing \in \mathcal{F}$. For the nontrivial case, let $A = (a_1, b_1] \cup \cdots \cup (a_n, b_n]$ with the constituent $(a_i, b_i]$'s denoting disjoint proper rsc subintervals of $(0,1]$. Then $A^c = (0, a_1] \cup (b_1, a_2] \cup \cdots \cup (b_n, 1]$, which is a finite disjoint union of rsc subintervals of Ω. [We will have $(0, a_1] = \varnothing$ and $(b_n, 1] = \varnothing$ when $a_1 = 0$ and $b_n = 1$.] Thus (F2) holds in all cases.

We now show that $\mathcal{F}$ is a π-system, which in conjunction with the above will show that $\mathcal{F}$ is a field. Let A and B be $\mathcal{F}$-sets. If A or B is empty, then $A \cap B = \varnothing \in \mathcal{F}$. We now deal with the nontrivial case, writing $A \in \mathcal{F}$ and $B \in \mathcal{F}$ as the disjoint unions $A = \bigcup_{i=1}^n (a_i, b_i]$ and $B = \bigcup_{j=1}^m (c_j, d_j]$. Thus $A \cap B = \bigcup_{i=1}^n \bigcup_{j=1}^m C_{ij}$, where each $C_{ij} = (a_i, b_i] \cap (c_j, d_j]$ and where both unions are disjoint. Each C_{ij} will either be a rsc subinterval of Ω or $\varnothing$, hence $A \cap B$ is either $\varnothing$ [hence in $\mathcal{F}$] or $A \cap B$ is a finite disjoint union of rsc subintervals of Ω [hence in $\mathcal{F}$]. Thus for every case we have $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$, and $\mathcal{F}$ is accordingly a π-system.

Were Ω any other rsc interval, then the collection consisting of $\varnothing$ and the finite disjoint unions of rsc subintervals of Ω would still be a field.

Example 4. Let $\Omega = \mathbb{R}$, and let $\mathcal{F}$ denote the collection consisting of $\varnothing$ and the finite disjoint unions of rsc intervals. $\mathcal{F}$ is not a field, for (F1) and (F2) fail.

Exercises.

1. The collection $\{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite}\}$ is a field on Ω.

2. Let $\mathcal{F} \subseteq 2^\Omega$ be such that $\Omega \in \mathcal{F}$ and $A - B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$. Then $\mathcal{F}$ is a field on Ω.

3. Every λ-system that is closed under *arbitrary* differences is a field.

4. Let $\mathcal{F} \subseteq 2^\Omega$ satisfy (F1) and (F2), and suppose that $\mathcal{F}$ is closed under finite *disjoint* unions. Then $\mathcal{F}$ is not necessarily a field.

5. Suppose that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots$, where $\mathcal{F}_n$ is a field on Ω for each $n \in \mathbb{N}$. Then $\bigcup_{n=1}^\infty \mathcal{F}_n$ is a field on Ω.

6. The collection consisting of $\mathbb{R}^k$, $\varnothing$, and all k-dimensional rectangles of all forms fails to be a field on $\mathbb{R}^k$.

7. The collection consisting of $\varnothing$ and the finite disjoint unions of k-dimensional rsc subrectangles of the given k-dimensional rsc rectangle $(\boldsymbol{a}, \boldsymbol{b}]$ is a field on Ω.

8*. An arbitrary intersection of fields on Ω is a field on Ω.

9*. Let Ω be arbitrary, and let $\mathcal{A} \subseteq 2^\Omega$. There exists a unique field $\mathcal{F}$ on Ω with the properties that (i) $\mathcal{A} \subseteq \mathcal{F}$, and (ii) if $\mathcal{G}$ is a field with $\mathcal{A} \subseteq \mathcal{G}$, then $\mathcal{F} \subseteq \mathcal{G}$. This field $\mathcal{F}$ is called the *[minimal] field [on Ω] generated by $\mathcal{A}$*. [2^Ω is a field on Ω that contains $\mathcal{A}$, hence a field containing $\mathcal{A}$ always exists. Let $\mathcal{F}$ denote the intersection of all fields on Ω that contain $\mathcal{A}$.]

10. Let $A_1, \cdots, A_n \subsetneq \Omega$ be disjoint. What does a typical element in the minimal field generated by $\{A_1, \cdots, A_n\}$ look like? [See Exercise 9.]

11. Let S be finite, and let Ω denote the set of sequences of elements of S. For each $\omega \in \Omega$, write $\omega = (z_1(\omega), z_2(\omega), \cdots)$, so that $z_k(\omega)$ denotes the kth term of ω for all $k \in \mathbb{N}$. For $n \in \mathbb{N}$ and $H \subseteq S^n$, let $C_n(H) = \{\omega \in \Omega : (z_1(\omega), \cdots, z_n(\omega)) \in H\}$. Let $\mathcal{F} = \{C_n(H) : n \in \mathbb{N},\ H \subseteq S^n\}$. Then $\mathcal{F}$ is a field of subsets of S^∞. [The sets $C_n(H)$ are called *cylinders of rank n*, and $\mathcal{F}$ is collection of all cylinders of all ranks.]

12*. Suppose that $\mathcal{A}$ is a semiring on Ω with $\Omega \in \mathcal{A}$. The collection of finite disjoint unions of $\mathcal{A}$-sets is a field on Ω. [Compare with Example 3 and Exercise 7.]

13*. Let $f : \Omega \to \Omega'$. Given $\mathcal{A}' \subseteq 2^{\Omega'}$, let $f^{-1}(\mathcal{A}') = \{f^{-1}(A') : A' \in \mathcal{A}'\}$, where $f^{-1}(A')$ is the usual inverse image of A' under f.
(a) If $\mathcal{A}'$ is a field on Ω', then $f^{-1}(\mathcal{A}')$ is a field on Ω.
(b) $f(\mathcal{A})$ may not be a field over Ω' even if $\mathcal{A}$ is a field on Ω.

14. Let Ω be infinite, and let $\mathcal{A} \subseteq 2^\Omega$ have cardinality $\aleph_0$. Let $f(\mathcal{A})$ denote the minimal field generated by $\mathcal{A}$ [Exercise 9]. We will show that $\text{card}(f(\mathcal{A})) = \aleph_0$.
(a) Given a collection $\mathcal{C}$, let $\mathcal{C}^*$ denote the collection of (i) finite unions of $\mathcal{C}$-sets, (ii) finite unions of differences of $\mathcal{C}$-sets, and (iii) finite unions of complements of $\mathcal{C}$-sets. If $\varnothing \in \mathcal{C}$, then $\mathcal{C} \subseteq \mathcal{C}^*$. If $\text{card}(\mathcal{C}) = \aleph_0$, then $\text{card}(\mathcal{C}^*) = \aleph_0$.
(b) Define $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_n = \mathcal{A}_{n-1}^*$ for $n \in \mathbb{N}$. Without any loss of generality we may assume that $\varnothing \in \mathcal{A}_0$ when considering the collection $\bigcup_{n=0}^\infty \mathcal{A}_n$.
(c) $\mathcal{A} \subseteq \bigcup_{n=0}^\infty \mathcal{A}_n \subseteq f(\mathcal{A})$, and $\text{card}(\bigcup_{n=0}^\infty \mathcal{A}_n) = \aleph_0$.
(d) $\bigcup_{n=0}^\infty \mathcal{A}_n$ is a field on Ω that contains $\mathcal{A}$.
(e) From (c) and (d), we have $f(\mathcal{A}) = \bigcup_{n=0}^\infty \mathcal{A}_n$, hence $\text{card}(f(\mathcal{A})) = \aleph_0$.
(f) We may generalize: if $\mathcal{A} \subseteq 2^\Omega$ is infinite, then $\text{card}(\mathcal{A}) = \text{card}(f(\mathcal{A}))$.

15. Some books work with a system of sets called an *algebra*. An *algebra on* Ω is a nonempty collection of subsets of Ω that satisfies (F2) and (F3).
(a) $\mathcal{F}$ is an algebra on Ω iff $\mathcal{F}$ is a ring on Ω with $\Omega \in \mathcal{F}$.
(b) $\mathcal{F}$ is an algebra iff $\mathcal{F}$ is a field. Thus *algebra* and *field* are synonymous.

1.3 σ-FIELDS

The most important set system is called a *σ-field*. Other authors might equivalently use the term *σ-algebra*. This set system will eventually be found in most of our work. A σ-field is a field with one extra assumption of a rather strong nature.

Definition. A collection $\mathcal{F}$ is a *σ-field on* Ω iff
(S1) $\Omega \in \mathcal{F}$,
(S2) $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$, and
(S3) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ whenever $A_1, A_2, \cdots \in \mathcal{F}$.

Closure properties (S1) and (S2) are exactly those for a field, but (S3) deals with closure under *countably infinite* unions, whereas property (F3) for fields dealt merely with closure under *finite* unions. From (S1) and (S2) it is automatic that $\varnothing$ is in every σ-field. From (S2) and (S3), a σ-field $\mathcal{F}$ is closed under countable intersections of $\mathcal{F}$-sets. Also, the collections $\{\varnothing, \Omega\}$ and 2^{Ω} are σ-fields, hence the concept of a σ-field is never vacuous.

Every σ-field is a field. To see this, let $\mathcal{F}$ denote a σ-field with $A, B \in \mathcal{F}$. Since $\varnothing \in \mathcal{F}$, we have $A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \cdots \in \mathcal{F}$ by (S3). Thus every σ-field is closed under finite unions as well. Similarly, every σ-field is also closed under finite intersections. Also, every finite field is a σ-field.

The main thing about σ-fields is that they are closed under the application of countably many of the standard set manipulations. The standard set operations are union, intersection, complementation, difference, and symmetric difference, and all of these can be expressed in terms of unions and complements. Thus, when one works with a collection of sets in a σ-field, one will never by using at most countably many set operations on these sets produce a set outside the σ-field.

Example 1. Let Ω be infinite, and let $\mathcal{F} = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite}\}$. It is trivial that $\mathcal{F}$ satisfies (S1) and (S2). To see that $\mathcal{F}$ is closed under union, let $A, B \in \mathcal{F}$. There are two cases: (i) both A and B finite, and (ii) at least one of A^c or B^c is finite. For (i) we have that $A \cup B$ is finite, whence $A \cup B \in \mathcal{F}$. For (ii), assume that B^c is finite. We have $(A \cup B)^c = A^c \cap B^c \subseteq B^c$, and thus $(A \cup B)^c$ is finite, so that again $A \cup B \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a field.

However, $\mathcal{F}$ fails to be a σ-field. To see this, pick a countably infinite subset $\{\omega_1, \omega_2, \cdots\}$ of Ω. Define $A_n = \{\omega_{2n}\}$ for $n \in \mathbb{N}$, hence $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. We have $\bigcup_{n=1}^{\infty} A_n = \{\omega_2, \omega_4, \omega_6 \cdots\}$, and $(\bigcup_{n=1}^{\infty} A_n)^c$ contains $\{\omega_1, \omega_3, \omega_5, \cdots\}$, which means that both $\bigcup_{n=1}^{\infty} A_n$ and $(\bigcup_{n=1}^{\infty} A_n)^c$ are infinite, giving $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{F}$. Thus, $\mathcal{F}$ fails to satisfy (S3). This gives a nontrivial example of a field that is not a σ-field.

Example 2. Let Ω be infinite, and let $\mathcal{F} = \{A \subseteq \Omega : A \text{ is amc or } A^c \text{ is amc}\}$. Again, (S1) and (S2) are trivially satisfied. To see (S3), let $\{A_n\}_{n=1}^{\infty}$ denote a sequence of $\mathcal{F}$-sets, and consider two cases: (i) each A_n is amc, and (ii) at least one A_n is such that A_n^c is amc. Since the countable union of amc sets is itself amc, (S3) holds in case (i). In case (ii), we assume without any loss of generality that A_1^c is amc, and we have that $(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c \subseteq A_1^c$, so that $(\bigcup_{n=1}^{\infty} A_n)^c$ is amc. It follows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ for case (ii) as well. Therefore, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ in both cases, so $\mathcal{F}$ is closed under countable unions, hence $\mathcal{F}$ is a σ-field.

Observe that if $\Omega = [0,1]$ and $U = [0, \frac{1}{2}]$, then both U and U^c are uncountable, whence $U \notin \mathcal{F}$. This shows that *a σ-field does not necessarily contain every subset of Ω*. Also, if we consider all singletons $\{x\}$ where $x \in U$, then for each $x \in U$ we have $\{x\} \in \mathcal{F}$ but $\bigcup_{x \in U}\{x\} = U \notin \mathcal{F}$, so that *a σ-field is not necessarily closed under completely arbitrary [uncountable] unions.*

The following claim is mimicked by Exercises 1.6, 1.12, 1.19(h), 2.8, and 2.9, and is of great importance for later devlopments. A simple fact used in the proof is this: *every arbitrary intersection of σ-fields on some common set Ω is itself a σ-field on Ω.* [The reader may easily verify this.]

Claim 1. *Let $\varnothing \neq \mathcal{A} \subseteq 2^{\Omega}$. There exists a unique σ-field $\sigma(\mathcal{A})$ on Ω such that (i) $\mathcal{A} \subseteq \sigma(\mathcal{A})$, and (ii) any σ-field $\mathcal{G}$ on Ω with $\mathcal{A} \subseteq \mathcal{G}$ is such that $\sigma(\mathcal{A}) \subseteq \mathcal{G}$.*

Proof: Let $\Sigma(\mathcal{A})$ denote the family of σ-fields on Ω that contain $\mathcal{A}$, and observe that $2^{\Omega} \in \Sigma(\mathcal{A})$, hence $\Sigma(\mathcal{A}) \neq \varnothing$. Let $\sigma(\mathcal{A}) = \bigcap_{\mathcal{F} \in \Sigma(\mathcal{A})} \mathcal{F}$. We have that $\sigma(\mathcal{A})$ is a σ-field since it is the intersection of σ-fields, and it is clear that $\mathcal{A} \subseteq \sigma(\mathcal{A})$. Therefore (i) holds. To verify (ii), if $\mathcal{G}$ is a σ-field on Ω that contains $\mathcal{A}$, then $\mathcal{G} \in \Sigma(\mathcal{A})$, hence $\sigma(\mathcal{A}) = \bigcap_{\mathcal{F} \in \Sigma(\mathcal{A})} \mathcal{F} \subseteq \mathcal{G}$.

All that remains is to show that $\sigma(\mathcal{A})$ is the unique σ-field on Ω with properties (i) and (ii). Suppose that $\mathcal{H}$ is a σ-field on Ω with (i) $\mathcal{A} \subseteq \mathcal{H}$ and (ii) any σ-field $\mathcal{G}$ with $\mathcal{A} \subseteq \mathcal{G}$ is such that $\mathcal{H} \subseteq \mathcal{G}$. By property (i) for $\sigma(\mathcal{A})$ and property (ii) for $\mathcal{H}$, we have that $\sigma(\mathcal{A}) \subseteq \mathcal{H}$. By property (ii) for $\sigma(\mathcal{A})$, we have $\mathcal{H} \subseteq \sigma(\mathcal{A})$. Therefore $\mathcal{H} = \sigma(\mathcal{A})$, and the entire proof is complete. ∎

Definition. We will call $\sigma(\mathcal{A})$ the *[minimal] σ-field [on Ω] generated by $\mathcal{A}$.*

Some properties are notable: $\varnothing \subsetneq \mathcal{A} \subseteq \mathcal{B} \subseteq 2^{\Omega}$ *implies* $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{B})$. To see this, observe that any σ-field over Ω that contains $\mathcal{B}$ must also contain $\mathcal{A}$, so $\Sigma(\mathcal{B}) \subseteq \Sigma(\mathcal{A})$. It follows that $\sigma(\mathcal{A}) = \bigcap_{\mathcal{F} \in \Sigma(\mathcal{A})} \mathcal{F} \subseteq \bigcap_{\mathcal{F} \in \Sigma(\mathcal{B})} \mathcal{F} = \sigma(\mathcal{B})$.

Next, *$\mathcal{A}$ is a σ-field iff $\mathcal{A} = \sigma(\mathcal{A})$*. To see this, let $\mathcal{A}$ denote a σ-field. By definition, $\mathcal{A} \subseteq \sigma(\mathcal{A})$; however, part (ii) of Claim 1 forces $\sigma(\mathcal{A}) \subseteq \mathcal{A}$, and hence $\mathcal{A} = \sigma(\mathcal{A})$. Conversely, if $\mathcal{A} = \sigma(\mathcal{A})$, then $\mathcal{A}$ is a σ-field since $\sigma(\mathcal{A})$ is.

Letting $\varnothing \subsetneq \mathcal{C} \subseteq 2^{\Omega}$, we take $\mathcal{A} = \sigma(\mathcal{C})$ in the previous paragraph. $\mathcal{A}$ is a σ-field over Ω, hence $\sigma(\mathcal{A}) = \mathcal{A}$, that is, $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$. Therefore, *$\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$ for any nonempty collection $\mathcal{C}$ of subsets of Ω.*

The concept of a minimal σ-field on Ω generated by a collection $\mathcal{A}$ may be extended to π-systems, λ-systems, and fields by the same reasoning exhibited in Claim 1. If $\varnothing \neq \mathcal{A} \subseteq 2^{\Omega}$, we let $\Pi(\mathcal{A})$, $\Lambda(\mathcal{A})$, and $F(\mathcal{A})$ denote the collections of all π-systems, λ-systems, and fields of subsets of Ω that contain $\mathcal{A}$, respectively. With this, we may define $\pi(\mathcal{A}) = \bigcap_{\mathcal{P}\in\Pi(\mathcal{A})} \mathcal{P}$, $\lambda(\mathcal{A}) = \bigcap_{\mathcal{L}\in\Lambda(\mathcal{A})} \mathcal{L}$, and $f(\mathcal{A}) = \bigcap_{\mathcal{F}\in F(\mathcal{A})} \mathcal{F}$, and we may show that these quantities are π-systems, λ-systems, and fields that contain $\mathcal{A}$. We may also show that if $\mathcal{G}$ is a π-system, λ-system, or field that contains $\mathcal{A}$, then $\pi(\mathcal{A})$, $\lambda(\mathcal{A})$, and $f(A)$ are contained in $\mathcal{G}$. Furthermore, each of $\pi(\mathcal{A})$, $\lambda(\mathcal{A})$, and $f(\mathcal{A})$ are the unique π-systems, λ-systems, and fields with such properties. Such demonstrations were the points of Exercises 1.6, 1.12, 2.8, and 2.9.

Finally, $\pi(\mathcal{A}) \subseteq f(\mathcal{A}) \subseteq \sigma(\mathcal{A})$ *and* $\pi(\mathcal{A}) \subseteq \lambda(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. To see this, observe that any σ-field containing $\mathcal{A}$ is a field containing $\mathcal{A}$, and any field containing $\mathcal{A}$ is a π-system containing $\mathcal{A}$, whence $\Sigma(\mathcal{A}) \subseteq F(\mathcal{A}) \subseteq \Pi(\mathcal{A})$, hence $\bigcap_{\mathcal{F}\in\Pi(\mathcal{A})} \mathcal{F} \subseteq \bigcap_{\mathcal{F}\in F(\mathcal{A})} \mathcal{F} \subseteq \bigcap_{\mathcal{F}\in\Sigma(\mathcal{A})} \mathcal{F}$, so $\pi(\mathcal{A}) \subseteq f(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. The other containment claim follows similarly and is left as an exercise.

The following result is important, and will be used later to characterize a certain collection of sets.

Claim 2. *Let $X : \Omega \to \Omega'$ be given with $\mathcal{A}' \subseteq 2^{\Omega'}$. Then (i) $X^{-1}(\sigma(\mathcal{A}'))$ is a σ-field on Ω, and (ii) we have $\sigma(X^{-1}(\mathcal{A}')) = X^{-1}(\sigma(\mathcal{A}'))$.*

Proof: We first show (i). Since $\Omega = X^{-1}(\Omega')$ and $\Omega' \in \sigma(\mathcal{A}')$, we have that $\Omega \in X^{-1}(\sigma(\mathcal{A}'))$. Next, if $A \in X^{-1}(\sigma(\mathcal{A}'))$, then $A = X^{-1}(B)$ for some $B \in \sigma(\mathcal{A}')$. Therefore, $A^c = X^{-1}(B)^c = X^{-1}(B^c)$, and $B^c \in \sigma(\mathcal{A}')$ since $\sigma(\mathcal{A}')$ is a σ-field. It follows that $A^c \in X^{-1}(\sigma(\mathcal{A}'))$, so that $X^{-1}(\sigma(\mathcal{A}'))$ is closed under complementation. To see that $X^{-1}(\sigma(\mathcal{A}'))$ is closed under countable unions, let $\{A_n\}_{n=1}^{\infty}$ denote a sequence of sets in $X^{-1}(\sigma(\mathcal{A}'))$. For each $n \in \mathbb{N}$, there is $B_n \in \sigma(\mathcal{A}')$ with $A_n = X^{-1}(B_n)$. Therefore $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} X^{-1}(B_n) = X^{-1}(\bigcup_{n=1}^{\infty} B_n)$, and $\bigcup_{n=1}^{\infty} B_n \in \sigma(\mathcal{A}')$ since $\sigma(\mathcal{A}')$ is a σ-field. It follows that $\bigcup_{n=1}^{\infty} A_n \in X^{-1}(\sigma(\mathcal{A}'))$, and thus $X^{-1}(\sigma(\mathcal{A}'))$ is closed under countable unions. Thus (i) stands proven.

We now turn to (ii). Since $\mathcal{A}' \subseteq \sigma(\mathcal{A}')$, we have $X^{-1}(\mathcal{A}') \subseteq X^{-1}(\sigma(\mathcal{A}'))$, and hence $\sigma(X^{-1}(\mathcal{A}')) \subseteq \sigma(X^{-1}(\sigma(\mathcal{A}'))) = X^{-1}(\sigma(\mathcal{A}'))$, where the equality follows from (i). This gives one containment relation. For the reverse containment relation, let $\mathcal{A}^* = \{A' \subseteq \Omega' : X^{-1}(A') \in \sigma(X^{-1}(\mathcal{A}'))\}$. It is direct to verify that (a) $\mathcal{A}^*$ contains $\mathcal{A}'$ and (b) $\mathcal{A}^*$ is a σ-field on Ω'. Since $\sigma(\mathcal{A}')$ is the smallest σ-field containing $\mathcal{A}'$, (a) and (b) yield $\sigma(\mathcal{A}') \subseteq \mathcal{A}^*$, so $X^{-1}(\sigma(\mathcal{A}')) \subseteq X^{-1}(\mathcal{A}^*) \subseteq \sigma(X^{-1}(\mathcal{A}'))$, where the last containment follows by the definition of $\mathcal{A}^*$. This gives the reverse containment relation, and the proof of (ii) is complete. ■

Exercises.

1*. A collection $\mathcal{F}$ of sets is called a *monotone class* iff (MC1) for every nondecreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{F}$-sets we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, and (MC2) for every nonincreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{F}$-sets we have $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

(a) If $\mathcal{F}$ is both a field and a monotone class, then $\mathcal{F}$ is a σ-field.

(b) A field is a monotone class if and only if it is a σ-field.

Monotone classes will be a main tool in discussing product measures and Fubini's Theorem in later chapters.

2*. This problem discusses some equivalent formulations of a σ-field.
(a) $\mathcal{F}$ satisfies (SF1), (SF2), and closure under amc intersections iff $\mathcal{F}$ is a σ-field.
(b) Every field that is closed under countable disjoint unions is a σ-field.
(c) If $\mathcal{F}$ satisfies (S1), closure under differences, and closure under countable unions or closure under countable intersections, then $\mathcal{F}$ is a σ-field.

3. Prove the following claims.
(a) A finite union of σ-fields on Ω is not necessarily a field on Ω.
(b) If a finite union of σ-fields on Ω is a field, then it is a σ-field as well.
(c) Given σ-fields $\mathcal{F}_1 \subsetneqq \mathcal{F}_2 \subsetneqq \cdots$ on Ω, it is not necessarily the case that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a σ-field. [Let $\Omega = \mathbb{N}$ and for all $n \in \mathbb{N}$ let $\mathcal{F}_n = \sigma(\{\{1\}, \cdots, \{n\}\})$.]

4*. Does Exercise 2.11 hold when *field* is replaced with *σ-field*?

5. A subset $A \subseteq \mathbb{R}$ is called *nowhere dense* iff every open interval I contains an open interval J such that $J \cap A = \varnothing$. Clearly $\varnothing$ and all subsets of a nowhere dense set are nowhere dense. A subset $A \subseteq \mathbb{R}$ is called a *set of the first category* iff A is a countable union of nowhere dense sets.
(a) An amc union of sets of the first category is of the first category.
(b) Let $\mathcal{F} = \{A \subseteq \mathbb{R} : A \text{ or } A^c \text{ is a set of the first category}\}$. Then $\mathcal{F}$ is a σ-field of subsets of $\mathbb{R}$. [To verify closure under countable unions, let $\{A_n\}_{n=1}^{\infty}$ denote a sequence of $\mathcal{F}$-sets. To show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, consider two cases: (i) each A_n is of the first category, and (ii) some A_n^c is of the first category.]

6. A *σ-ring of subsets of* Ω is a nonempty collection of subsets of Ω that is closed under differences as well as countable unions.
(a) Every σ-ring is closed under finite unions and amc intersections.
(b) $\mathcal{F}$ is a σ-field iff $\mathcal{F}$ is a σ-ring with $\Omega \in \mathcal{F}$.
(c) State and prove an existence and uniqueness result regarding the *[minimal] σ-ring generated by a collection $\mathcal{A}$ of subsets of Ω.*

7. This exercise continues Exercise 6. Let $\mathcal{A}$ denote a collection of subsets of some set Ω, and let $S(\mathcal{A})$ denote the minimal σ-ring that contains $\mathcal{A}$.
(a) For any collection $\mathcal{C}$ of sets, let $\mathcal{C}^*$ denote the collection of all countable unions of differences of $\mathcal{C}$-sets. Define $\mathcal{A}_0 = \mathcal{A}$, and for any ordinal $\alpha \succ 0$, define $\mathcal{A}_\alpha = (\bigcup_{0 \preceq \beta \prec \alpha} \mathcal{A}_\beta)^*$. Then $0 \preceq \alpha \prec \beta$ implies $\mathcal{A} \subseteq \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \subseteq S(\mathcal{A})$.
(b) Letting ω denote the first uncountable ordinal, show that $S(\mathcal{A}) = \bigcup_{0 \preceq \alpha \prec \omega} \mathcal{A}_\alpha$. [Hint: for any sequence of ordinals $\{\alpha_n\}_{n=1}^{\infty}$ with each $\alpha_n \prec \omega$, there exists an ordinal $\gamma \prec \omega$ with $\alpha_n \prec \gamma$ for all $n \in \mathbb{N}$.]
(c) If $\text{card}(\mathcal{A}) \leq \mathfrak{c}$, then $\text{card}(S(\mathcal{A})) \leq \mathfrak{c}$. [Use the fact that a union of continuum-many sets that are amc must have cardinality no greater than $\mathfrak{c}$.]

8. We will show that no σ-field can ever be countably infinite.
(a) If Ω is finite, then any σ-field on Ω is finite.
(b) For the remainder, assume that Ω is infinite and that there is a countably infinite σ-field $\mathcal{F} = \{A_1, A_2, \cdots\}$ on Ω. For each $\omega \in \Omega$, let $B_\omega = \bigcap\{A_n : \omega \in A_n\}$. For any distinct $\omega, \omega' \in \Omega$, either $B_\omega = B_{\omega'}$ or $B_\omega \cap B_{\omega'} = \varnothing$.
(c) From (b), there exists a disjoint collection $\{C_1, C_2, \cdots\}$ of nonempty $\mathcal{F}$-sets with $\bigcup_{n=1}^{\infty} C_n = \Omega$.

(d) For each sequence $e = (e_1, e_2, \cdots)$ of 0's and 1's, let $D_e = \bigcup\{C_n : e_n = 1\}$. Then $e \neq e'$ forces $D_e \neq D_{e'}$, and a contradiction ensues. Hence, there cannot exist a countably infinite σ-field on our infinite Ω.

9*. This problem is meant to give practice for some simple claims.
(a) If $\mathcal{A} \subseteq \mathcal{A}' \subseteq \sigma(\mathcal{A})$, then $\sigma(\mathcal{A}') = \sigma(\mathcal{A})$.
(b) For any collection $\varnothing \neq \mathcal{A} \subseteq 2^\Omega$, $\pi(\mathcal{A}) \subseteq \lambda(\mathcal{A}) \subseteq \sigma(\mathcal{A})$.
(c) If the nonempty collection $\mathcal{A}$ is finite, then $\sigma(\mathcal{A}) = f(\mathcal{A})$.
(d) For arbitrary collections $\mathcal{A}$, we have $\sigma(\mathcal{A}) = \sigma(f(\mathcal{A}))$.
(e) For arbitrary collections $\mathcal{A}$, we have $f(\sigma(\mathcal{A})) = \sigma(f(\mathcal{A}))$.

10. We will show that $f(\mathcal{A})$ is the class of sets of the form $\bigcup_{i=1}^m \bigcap_{j=1}^{n_i} A_{ij}$, where for each (i,j) pair either A_{ij} or A_{ij}^c is in $\mathcal{A}$, and where $\bigcap_{j=1}^{n_1} A_{1j}, \cdots, \bigcap_{j=1}^{n_m} A_{mj}$ are disjoint.
(a) Let $\mathcal{C}$ denote the class of sets of the stated form. Then $\Omega \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersection.
(b) Show that $\mathcal{C}$ is closed under complementation by first showing that

$$\left(\bigcup_{i=1}^m \bigcap_{j=1}^{n_i} A_{ij}\right)^c = \bigcap_{i=1}^m \left(A_{i1}^c \cup \bigcup_{j=2}^{n_i} \left(A_{ij}^c \cap \bigcap_{k=1}^{j-1} A_{ik}\right)\right),$$

where if $n_i < 2$, $\bigcup_{j=2}^{n_i} \left(A_{ij}^c \cap \bigcap_{k=1}^{j-1} A_{ik}\right)$ is defined as $\varnothing$, $i = 1, \cdots, m$.
(c) For $i = 1, \cdots, m$, $\bigcup_{j=2}^{n_i} \left(A_{ij}^c \cap \bigcap_{k=1}^{j-1} A_{ik}\right) \in \mathcal{C}$. Also, by closure under intersection, we have $\left(\bigcup_{i=1}^m \bigcap_{j=1}^{n_i} A_{ij}\right)^c \in \mathcal{C}$.
(d) $\mathcal{C}$ is a field that contains $\mathcal{A}$, and thus $f(\mathcal{A}) \subseteq \mathcal{C}$.
(e) Every $\mathcal{C}$-set must be in any field containing $\mathcal{A}$, hence $\mathcal{C} \subseteq f(\mathcal{A})$.

11. If $\text{card}(\mathcal{A}) = \aleph_0$, then $\text{card}(f(\mathcal{A})) = \aleph_0$. [Use Exercise 10.]

12. When $\mathcal{A} = \{A_1, \cdots, A_n\}$, $\sigma(\mathcal{A})$ has at most 2^{2^n} sets. The bound is achieved if $A_1, \cdots, A_n$ are disjoint.

13. Let $\mathcal{A} = \{\{\omega\} : \omega \in \Omega\}$, where Ω as usual denotes a nonempty set.
(a) $f(\mathcal{A}) = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite}\}$.
(b) When Ω is amc, $\sigma(\mathcal{A}) = 2^\Omega$.
(c) For general Ω, $\sigma(\mathcal{A}) = \{A \subseteq \otimes : A \text{ or } A^c \text{ is amc}\}$.

14. Suppose that $\{\mathcal{F}_n\}_{n=1}^\infty$ is a sequence of σ-fields on Ω with the property that each $\mathcal{F}_n$ is generated by a countable collection of subsets of Ω. Then the minimal σ-field containing each of $\mathcal{F}_1, \mathcal{F}_2, \cdots$, that is, the intersection of all σ-fields on Ω that contain each of $\mathcal{F}_1, \mathcal{F}_2, \cdots$, is generated by an amc collection of subsets of Ω.

15. Let $\mathcal{F}$ denote any σ-field on Ω other than 2^Ω, and let $H \subsetneq \Omega$ be such that $H \notin \mathcal{F}$. Then $\sigma(\mathcal{F} \cup \{H\}) = \{(H \cap A) \cup (H^c \cap B) : A, B \in \mathcal{F}\}$. [Let $\mathcal{C}$ denote the sets of the form $(H \cap A) \cup (H^c \cap B)$, where A and B range over $\mathcal{F}$. One handily obtains $\sigma(\mathcal{F} \cup \{H\}) \supseteq \mathcal{C}$. The reverse inclusion is more difficult. Argue that $\mathcal{C}$ is a σ-field of subsets. To show (S2), argue that $(A \cup B)^c \subseteq (H \cap A^c) \cup (H^c \cap B^c)$. Show that $\mathcal{F} \cup \{H\} \subseteq \mathcal{C}$, and invoke minimality.]

16. Let $\mathcal{F} = \sigma(\mathcal{A})$, where $\varnothing \neq \mathcal{A} \subseteq 2^\Omega$. For each $B \in \mathcal{F}$ there exists a countable subcollection $\mathcal{A}_B \subseteq \mathcal{A}$ with $B \in \sigma(\mathcal{A}_B)$. [Let $\mathcal{B}$ to be the set of all $B \in \mathcal{F}$ with this property. Thus $\mathcal{B} \subseteq \mathcal{F}$. Next, show that $\mathcal{B}$ is itself a σ-field containing each $\mathcal{A}$-set, so that $\mathcal{F} \subseteq \mathcal{B}$.]

17. The following parts are of a similar flavor.
(a) Suppose that $\mathcal{A}$ is such that $\sigma(\mathcal{A}) = 2^\Omega$. For each distinct pair $\omega, \omega' \in \Omega$ there exists $A \in \mathcal{A}$ with $\omega \in A$ but $\omega' \notin A$.
(b) Let Ω be countable, and suppose that for each distinct pair $\omega, \omega' \in \Omega$, there exists a set $A \in \mathcal{A}$ such that $\omega \in A$ but $\omega' \notin A$. Then $\sigma(\mathcal{A}) = 2^\Omega$.

18. Given $\varnothing \neq \mathcal{A} \subseteq 2^\Omega$ and $\varnothing \neq B \subseteq \Omega$, let $\mathcal{A} \cap B = \{A \cap B : A \in \mathcal{A}\}$ and let $\sigma(\mathcal{A}) \cap B = \{A \cap B : A \in \sigma(\mathcal{A})\}$.
(a) $\sigma(\mathcal{A}) \cap B$ is a σ-field on B.
(b) Next, define $\sigma_B(\mathcal{A} \cap B)$ to be the minimal σ-field over B generated by the class $\mathcal{A} \cap B$. Then $\sigma_B(\mathcal{A} \cap B) = \sigma(\mathcal{A}) \cap B$. [Use (a) to argue that $\sigma_B(\mathcal{A} \cap B) \subseteq \sigma(\mathcal{A}) \cap B$. Next, let $\mathcal{C} = \{A \in \sigma(\mathcal{A}) : A \cap B \in \sigma_B(\mathcal{A} \cap B)\} \subseteq \sigma(\mathcal{A})$. Verify that $\mathcal{C}$ is a σ-field that contains $\mathcal{A}$, hence $\sigma(\mathcal{A}) = \mathcal{C}$. This will give $\sigma(\mathcal{A}) \cap B \subseteq \sigma_B(\mathcal{A} \cap B)$.]

19*. Suppose that $\mathcal{A} = \{A_1, A_2, \cdots\}$ is a disjoint sequence of subsets of Ω with $\bigcup_{n=1}^\infty A_n = \Omega$. Then each $\sigma(\mathcal{A})$-set is the union of an at most countable subcollection of $A_1, A_2, \cdots$. [Define $\mathcal{C}$ as the class of $A \in \sigma(\mathcal{A})$ such that A is an at most countable union of $\mathcal{A}$-sets. Show that $\mathcal{C}$ is a σ-field of subsets of Ω with $\mathcal{C} \supseteq \mathcal{A}$, which gives $\mathcal{C} = \sigma(\mathcal{A})$.]

20*. Let $\mathcal{P}$ denote a π-system on Ω, and let $\mathcal{L}$ denote a λ-system on Ω with $\mathcal{P} \subseteq \mathcal{L}$. We will show that $\sigma(\mathcal{P}) \subseteq \mathcal{L}$. Let $\lambda(\mathcal{P})$ denote the λ-system generated by $\mathcal{P}$, and for each subset $A \subseteq \Omega$ we define $\mathcal{G}_A = \{C \subseteq \Omega : A \cap C \in \lambda(\mathcal{P})\}$.
(a) $\mathcal{P} \subseteq \lambda(\mathcal{P}) \subseteq \mathcal{L}$.
(b) If $\lambda(\mathcal{P})$ is shown to be a π-system, then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.
(c) For any $A \in \mathcal{P}$, $\mathcal{G}_A$ is a λ-system containing $\mathcal{P}$. If $A \in \lambda(\mathcal{P})$, then $\mathcal{G}_A$ is a λ-system.
(d) For all $A \in \mathcal{P}$, $\lambda(\mathcal{P}) \subseteq \mathcal{G}_A$.
(e) For all $A \in \mathcal{P}$ and $B \in \lambda(\mathcal{P})$ we have $A \in \mathcal{G}_B$.
(f) For all $B \in \lambda(\mathcal{P})$, we have $\lambda(\mathcal{P}) \subseteq \mathcal{G}_B$.
(g) $\lambda(\mathcal{P})$ is a π-system, hence $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Comment: This result is called *Dynkin's π-λ Theorem*, and it is a tool used in proving a uniqueness question regarding measures in Chapter 3. All parts of this exercise should be straightforward.

21*. Let $\mathcal{F}$ denote a field on Ω, and let $\mathcal{M}$ denote a monotone class on Ω [See Exercise 1]. We will show that $\mathcal{F} \subseteq \mathcal{M}$ implies $\sigma(\mathcal{F}) \subseteq \mathcal{M}$. Let $m(\mathcal{F})$ denote the *minimal monotone class on Ω generated by $\mathcal{F}$*. That is, $m(\mathcal{F})$ is the intersection of all monotone classes on Ω containing the collection $\mathcal{F}$.
(a) To prove the claim, it is sufficient to show that $\sigma(\mathcal{F}) \subseteq m(\mathcal{F})$.
(b) If $m(\mathcal{F})$ is a field, then $\sigma(\mathcal{F}) \subseteq m(\mathcal{F})$.
(c) $\Omega \in m(\mathcal{F})$.
(d) Let $\mathcal{G} = \{A \subseteq \Omega : A^c \in m(\mathcal{F})\}$. $\mathcal{G}$ is a monotone class on Ω and $m(\mathcal{F}) \subseteq \mathcal{G}$.
(e) $m(\mathcal{F})$ is indeed closed under complementation.
(f) Let $\mathcal{G}_1 = \{A \subseteq \Omega : A \cup B \in m(\mathcal{F}) \text{ for all } B \in \mathcal{F}\}$. Then $\mathcal{G}_1$ is a monotone class such that $\mathcal{F} \subseteq \mathcal{G}_1$ and $m(\mathcal{F}) \subseteq \mathcal{G}_1$.
(g) Let $\mathcal{G}_2 = \{B \subseteq \Omega : A \cup B \in m(\mathcal{F}) \text{ for all } A \in m(\mathcal{F})\}$. Then $\mathcal{G}_2$ is a monotone class such that $\mathcal{F} \subseteq \mathcal{G}_2$, and $m(\mathcal{F}) \subseteq \mathcal{G}_2$.
(h) $m(\mathcal{F})$ is closed under finite unions, and hence is a field.

Comment: This result is called *Halmos' Monotone Class Theorem*, and is similar to the π-λ *Theorem* above.

1.4 THE BOREL σ-FIELD

This and the next section deal with the two very important special cases of the abstract theory regarding minimal σ-fields presented in the previous section. The definition and discussions follow.

Definition. The *Borel σ-field [on $\mathbb{R}$]* is denoted by $\mathcal{B}$ and is defined as $\sigma(\{(a,b] : -\infty < a < b < +\infty\})$. The sets in $\mathcal{B}$ are called *[linear, one-dimensional] Borel sets.*

The concepts that make $\mathcal{B}$ worth careful study are not at all obvious, and there seems to be nothing so intrinsically special about the rsc subintervals of $\mathbb{R}$ that the σ-field on $\mathbb{R}$ generated by them is noteworthy. An explanation of the significance of $\mathcal{B}$ is to be found in Chapter 3 and more so in Chapter 4, when Lebesgue measure is discussed.

Notation. We will use the following notation again and again in this section, and it is best to define everything in one place. Here, $x, a, b \in \mathbb{R}$ with $a < b$.

$\mathcal{A}_1$ = intervals of the form $(-\infty, x]$	$\mathcal{A}_2$ = intervals of the form $(-\infty, x)$
$\mathcal{A}_3$ = intervals of the form $[x, \infty)$	$\mathcal{A}_4$ = intervals of the form (x, ∞)
$\mathcal{A}_5$ = intervals of the form $[a, b]$	$\mathcal{A}_6$ = intervals of the form $(a, b]$
$\mathcal{A}_7$ = intervals of the form $[a, b)$	$\mathcal{A}_8$ = intervals of the form (a, b)
$\mathcal{A}_9$ = open subsets of $\mathbb{R}$	$\mathcal{A}_{10}$ = closed subsets of $\mathbb{R}$.

For $n = 1, \cdots, 8$, $\mathcal{A}_n^*$ will denote the collection of intervals having the same form as those in $\mathcal{A}_n$, except that the endpoints are rational. For example, $\mathcal{A}_6^*$ denotes the set of intervals of the form $(a, b]$ with $a, b \in \mathbb{Q}$, $\mathcal{A}_8^*$ denotes the set of open intervals (a, b) with $a, b \in \mathbb{Q}$, etc.

In the definition above, $\mathcal{B}$ was defined as $\sigma(\mathcal{A}_6)$, and it might be wondered just why $\sigma(\mathcal{A}_6)$ is of importance instead of [say] $\sigma(\mathcal{A}_3)$ or any of the other σ-fields generated by $\mathcal{A}_n$, $n \neq 3, 6$. It will be seen below that the choice of $\mathcal{A}_6$ for a generating class as compared to [say] $\mathcal{A}_3$ or [say] $\mathcal{A}_8^*$ is wholly arbitrary, as any of $\mathcal{A}_1, \cdots, \mathcal{A}_{10}, \mathcal{A}_1^*, \cdots, \mathcal{A}_8^*$ generates $\mathcal{B}$. This result is important to know because various books define $\mathcal{B}$ differently. Some define $\mathcal{B}$ as $\sigma(\mathcal{A}_6)$, while some write $\mathcal{B} = \sigma(\mathcal{A}_9)$. Halmos' text on measure theory writes $\mathcal{B} = \sigma(\mathcal{A}_7)$, etc. We will show that all of these definitions are equivalent, so that these seemingly different definitions are all ultimately saying the same thing.

Claim 1. *Let $\mathcal{C}, \mathcal{D} \subseteq 2^\Omega$ each be nonempty. If each $\mathcal{C}$-set may be written in terms of complements of $\mathcal{D}$-sets, amc unions of $\mathcal{D}$-sets, and amc intersections of $\mathcal{D}$-sets, then $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{D})$. If in addition each $\mathcal{D}$-set may be written similarly in terms of $\mathcal{C}$-sets, then $\sigma(\mathcal{C}) = \sigma(\mathcal{D})$.*

Proof: The hypotheses imply that each $\mathcal{C}$-set will be in every σ-field containing $\mathcal{D}$, for σ-fields are closed under complements, amc unions, amc intersections, and any amc sequences of such operations. Therefore, each $\mathcal{C}$-set is in the intersection of each σ-field containing $\mathcal{D}$: $\mathcal{C} \subseteq \sigma(\mathcal{D})$. Thus $\sigma(\mathcal{C}) \subseteq \sigma(\sigma(\mathcal{D})) = \sigma(\mathcal{D})$. Under the additional assumptions of the claim, the reverse inclusion $\sigma(\mathcal{D}) \subseteq \sigma(\mathcal{C})$ follows in the exact same fashion. ∎

Claim 2. $\mathcal{B} = \sigma(\mathcal{A}_1) = \cdots = \sigma(\mathcal{A}_{10}) = \sigma(\mathcal{A}_1^*) = \cdots = \sigma(\mathcal{A}_8^*)$.

Proof: Let $x, a, b \in \mathbb{R}$ with $a < b$. Letting the unions and intersections below be taken over all $n \in \mathbb{N}$, it may be verified that

$$(-\infty, x) = \bigcup(-\infty, x - n^{-1}], \qquad [x, +\infty) = (-\infty, x)^c,$$

$$(x, +\infty) = \bigcup[x + n^{-1}, +\infty), \qquad [a, b] = (b, +\infty)^c - \left[\bigcup(a - n^{-1}, +\infty)\right]^c,$$

$$(a, b] = \bigcup[a + n^{-1}, b], \qquad [a, b) = \left((a, b] \cup \bigcap(a - n^{-1}, a]\right) - \bigcap(b - n^{-1}, b],$$

$$\text{and} \quad (a, b) = \bigcup[a + n^{-1}, b).$$

Recall next that every open subset of $\mathbb{R}$ is an amc union of sets of the form (a, b), every closed subset of $\mathbb{R}$ is the complement of an open subset of $\mathbb{R}$, and $(a, b] = [(-\infty, a] \cup \bigcup[b + n^{-1}, +\infty)]^c$, which is the complement of a countable union of closed sets. Hence, for $n = 2, \cdots, 10$, we have written a typical element of $\mathcal{A}_n$ in terms of amc unions, amc intersections, and complements of sets in $\mathcal{A}_{n-1}$, and $\mathcal{A}_1$ has been similarly expressed in terms of $\mathcal{A}_{10}$-sets. We may therefore invoke Claim 1 to obtain $\sigma(\mathcal{A}_1) \subseteq \cdots \subseteq \sigma(\mathcal{A}_{10}) \subseteq \sigma(\mathcal{A}_1)$, and thus these inclusions are actually equalities.

Next, taking $x, a, b \in \mathbb{Q}$ above, noting that $(-\infty, x] = (\bigcup(x, b + n))^c$, and using Claim 1 yields $\sigma(\mathcal{A}_1^*) \subseteq \cdots \subseteq \sigma(\mathcal{A}_8^*) \subseteq \sigma(\mathcal{A}_1^*)$, and hence these inclusions are actually equalities. We thus have $\sigma(\mathcal{A}_1) = \cdots = \sigma(\mathcal{A}_{10})$ and $\sigma(\mathcal{A}_1^*) = \cdots = \sigma(\mathcal{A}_8^*)$.

Finally, observe that for any $x \in \mathbb{R}$ we may find an increasing rational sequence $\{r_n\}_{n=1}^{\infty}$ with limit x. Since $(-\infty, x) = \bigcup(-\infty, r_n)$, elements of $\mathcal{A}_2$ can be expressed in terms of amc unions of $\mathcal{A}_2^*$-sets, whence $\sigma(\mathcal{A}_2) \subseteq \sigma(\mathcal{A}_2^*)$. But $\mathcal{A}_2^* \subseteq \mathcal{A}_2$, hence $\sigma(\mathcal{A}_2^*) \subseteq \sigma(\mathcal{A}_2)$, ergo $\sigma(\mathcal{A}_2) = \sigma(\mathcal{A}_2^*)$, which, with the other equalites, finishes the proof. ∎

Since $\mathcal{A} \subseteq \sigma(\mathcal{A})$ for arbitrary $\mathcal{A}$, the above theorem says that $\mathcal{B}$ contains $\mathcal{A}_1, \cdots, \mathcal{A}_{10}$. That is, $\mathcal{B}$ contains every interval of every form, every open set, and every closed set. Furthermore, $\mathcal{B}$ contains any set that can be obtained from the sets in these classes via amc combinations of amc unions, amc intersections, and complements. It will eventually turn out that $\mathcal{B}$ will contain just about every subset of $\mathbb{R}$ that one can imagine, although it will also turn out that (i) there are *many* subsets of $\mathbb{R}$ that are *not* in $\mathcal{B}$, and (ii) there are many sets in $\mathcal{B}$ that are inexpressible in terms of amc combinations of amc unions, amc intersections, and complements of sets in any and all of the 10 aforementioned classes. Assertions (i) and (ii) are given rigorous proofs in Sections 1.6 and 1.7.

A few extra questions of a somewhat courageous nature might have suggested themselves by now. These questions are now listed in no special order.

(Q1) *What is* card$(\mathcal{B})$*? That is, "how many" Borel sets are there?*

(Q2) *What does a generic Borel set "look like?"*

(Q3) *Is there some type of "constructive algorithm" for generating $\mathcal{B}$ from the sets in some or all of the $\mathcal{A}_1, \cdots, \mathcal{A}_{10}, \mathcal{A}_1^*, \cdots, \mathcal{A}_8^*$?*

(Q4) *Why look at the minimal σ-field generated by any of the classes in Claim 2 when one could have the largest σ-field simply by dealing with $2^{\mathbb{R}}$?*

Such questions actually invoke some of the very interesting and foundational items from modern set theory and analysis, such as the Axiom of Choice[1] (AC) and the various forms of the Continuum Hypothesis[2] (CH). We will take both of these items as true in this text, and shall attempt loose sketches of the answers to questions (Q1)-(Q4) as posed.

Since card$(\mathcal{A}_6) = \mathfrak{c}$ and $\mathcal{A}_6 \subseteq \mathcal{B}$, we have card$(\mathcal{B}) \geq \mathfrak{c}$. Since $\mathcal{B} \subseteq 2^{\mathbb{R}}$ and card$(2^{\mathbb{R}}) = 2^{\mathfrak{c}}$, we have card$(\mathcal{B}) \leq 2^{\mathfrak{c}}$. Therefore $\mathfrak{c} \leq$ card$(\mathcal{B}) \leq 2^{\mathfrak{c}}$. If we adopt the Generalized Continuum Hypothesis[3] (GCH), this fact forces either card$(\mathcal{B}) = \mathfrak{c}$ or card$(\mathcal{B}) = 2^{\mathfrak{c}}$. It is not obvious which possibility is true, for both seem plausible. In Section 1.6 we rigorously show that card$(\mathcal{B}) = \mathfrak{c}$. If we accept this as true, then card$(2^{\mathbb{R}}) = 2^{\mathfrak{c}} > \mathfrak{c} =$ card$(\mathcal{B})$, so that there are subsets of $\mathbb{R}$ outside $\mathcal{B}$. In fact, since $2^{\mathfrak{c}} - \mathfrak{c} = 2^{\mathfrak{c}}$, "most" subsets of $\mathbb{R}$ are not Borel sets. In Chapter 4, we will use (AC) to exhibit as explicitly as possible some sets outside $\mathcal{B}$.

The answers to (Q2) and (Q3) require substantial work. For now, the answer to (Q2) is that there is no "nice" generic representation formula for an arbitrary $B \in \mathcal{B}$. One might very well think that a typical $B \in \mathcal{B}$ might have the form $B = A_1 \star A_2 \star A_3 \star \cdots$, where each $\star$ can freely denote $\cup$ or $\cap$, and where each A_n freely denotes an interval of any form, a singleton, a finite set, an open set, a closed set, a countable set, or a complement of any one of these types of "nice" sets. However, this scheme, while accounting for a subset of $\mathcal{B}$ with cardinality $\mathfrak{c}$, will *not* be sufficient to account for *all* Borel sets. In fact,

[1]One form of (AC) is as follows: Given any collection of nonempty sets $\mathcal{A} = \{A_i : i \in I\}$, there exists a function ϕ with domain $\mathcal{A}$ and the property that $\phi(A_i) \in A_i$ for each $i \in I$. More loosely put, given an arbitrary family of nonempty sets, we can form a new set consisting of one element from each set in the family.

[2]One form of (CH) is this: There exists no cardinal number strictly between $\aleph_0$ and $\mathfrak{c}$. In advanced set theory, it is shown that (i) both (CH) and (AC) are independent of the usual Zermelo-Fraenkel (ZF) axioms, and (ii) in the presence of (ZF), (CH) and (AC) are independent.

[3](GCH) asserts that for any infinite cardinal $\mathfrak{u}$ there is no cardinal $\mathfrak{v}$ with $\mathfrak{u} < \mathfrak{v} < 2^{\mathfrak{u}}$. In advanced set theory, it is shown that (GCH) implies both (AC) and (CH), so we are forced to adopt (GCH) to pursue this discussion.

the collection of all sets generated by this scheme will be a "small" proper subset of $\mathcal{B}$. One might think that sets of the form $B = B_1 \star B_2 \star B_3 \star \cdots$, where each B_n is a set like $A_1 \star A_2 \star \cdots$ in the previous formula, would account for all of the Borel sets. Again, $\mathfrak{c}$ more Borel sets will be accounted for by this more comprehensive representation, but many will be excluded. One might iterate this process countably many times, obtaining $\aleph_0 \mathfrak{c} = \mathfrak{c}$ Borel sets, but surprisingly there will still be $\mathfrak{c}$ Borel sets that are excluded. [See Section 1.7.] Thus, in proving things about $\mathcal{B}$, we cannot take a generic $B \in \mathcal{B}$ and invoke a "nice" representation of B, using this hoped-for representation to prove a property about our generic B and hence about all Borel sets. Any argument involving $\mathcal{B}$ will be "nonconstructive" in the same sense that many of the exercises in the previous section were.

The answer to (Q3) is in the negative if we want an algorithm that is "practical" or not too esoteric. In imprecise language, it will be shown that we cannot arrive at all of $\mathcal{B}$ by taking all intervals of all forms and employing a recursive scheme that uses countably many operations "in a given order" at each stage, no matter how many stages we let the scheme run. On the other hand, there *is* a scheme that uses countably many operations "not performed in a simple sequence" at each stage that "after countably many stages" will produce $\mathcal{B}$; the discussion of such a scheme requires using ordinal numbers. These utterly imprecise answers would make a professional set theorist or logician chafe, and it might be wondered just what all of these loosely worded answers really mean. Perhaps it is best to just say that (Q3) has a negative answer, and a rigorous formulation of (Q3) is found in Section 1.6.

The answer to (Q4) will be completely answered in a later chapter. It suffices to say for the present that $\mathcal{B}$ is, in a certain vague sense, the "largest σ-field containing all of the 'ordinary' sets from analysis that will not cause mathematical and logical problems with (AC) and our upcoming definition of what will be called Lebesgue measure." This very nebulous statement will be completely demystified in the proofs of the two "impossibility theorems" found at the end of Chapter 4.

The final topic of this section concerns the *extended Borel σ-field on* $\bar{\mathbb{R}}$, and it is a simple extension of what has been discussed regarding $\mathcal{B}$ and $\mathbb{R}$.

Definition. The *extended Borel σ-field on* $\bar{\mathbb{R}}$ is denoted by $\bar{\mathcal{B}}$, and is defined by $\bar{\mathcal{B}} \boxminus \sigma(\{(a,b] : a,b \in \bar{\mathbb{R}}, a < b\})$. Sets in $\bar{\mathcal{B}}$ are called *extended Borel sets.*

Just as different authors use different definitions for $\mathcal{B}$, so too $\bar{\mathcal{B}}$ has various definitions. Some other definitions are (1) the class of sets that are $\mathcal{B}$-sets or are $\mathcal{B}$-sets enlarged by one or both of $\pm\infty$, (2) the σ-field on $\bar{\mathbb{R}}$ generated by the class of intervals of the form (a,b), $[-\infty, a)$, and $(b, +\infty]$, where $a, b \in \mathbb{R}$, (3) the σ-field on $\bar{\mathbb{R}}$ generated by intervals of the form $[-\infty, x)$, $x \in \mathbb{R}$, (4) $\sigma(\mathcal{B} \cup \{\{-\infty\}, \{+\infty\}\})$, and (5) the σ-field on $\bar{\mathbb{R}}$ generated by the open subsets of $\bar{\mathbb{R}}$, sets of the form A, $A \cup (x, \infty]$, $A \cup [-\infty, x)$, and $A \cup [-\infty, x) \cup (y, +\infty]$, where A is an open subset of $\mathbb{R}$ and $x, y \in \mathbb{R}$. The reader may rest assured that all of these definitions are equivalent to ours. In fact, (1) may be the most useful working characterization of $\bar{\mathcal{B}}$-sets. It is straightforward to verify the equivalences; see Exercise 6.

There are some final points that should be stated. First, *every Borel set is an extended Borel set.* Second, *the definition of $\bar{\mathcal{B}}$ implies that $\{-\infty, +\infty\}$, $\{+\infty\}$, and $\{-\infty\}$ are extended Borel sets.* These facts are obtained by using (1): If every $\bar{\mathcal{B}}$-set is a $\mathcal{B}$-set possibly augmented by one or both points $\pm\infty$, then, taking $\varnothing \in \mathcal{B}$, it follows that $\varnothing \cup \{-\infty, +\infty\}$, $\varnothing \cup \{-\infty\}$, and $\varnothing \cup \{+\infty\}$ are $\bar{\mathcal{B}}$-sets. Third, the answers to questions (Q1)-(Q4) are not changed in any way by now allowing consideration of $\pm\infty$. Finally, the natural extension of Claim 2 holds in the extended Borel setting.

Exercises.

1*. Show directly that $\sigma(\mathcal{A}_3) = \sigma(\mathcal{A}_3^*)$, $\sigma(\mathcal{A}_4) = \sigma(\mathcal{A}_7)$, and $\sigma(\mathcal{A}_4^*) = \sigma(\mathcal{A}_{10})$.

2*. All amc subsets of $\mathbb{R}$ are Borel sets. All subsets of $\mathbb{R}$ that differ from a Borel set by at most countably many points are Borel sets. That is, if the symmetric difference $C \triangle B$ is amc and $B \in \mathcal{B}$, then $C \in \mathcal{B}$.

3*. The *Borel σ-field on* $(0, 1]$ is denoted by $\mathcal{B}_{(0,1]}$ and is defined as the σ-field on $(0, 1]$ generated by the rsc subintervals of $(0, 1]$. $\mathcal{B}_{(0,1]}$ may be equivalently defined by $\{B \cap (0, 1] : B \in \mathcal{B}\}$.

4*. $\mathcal{B}$ is generated by the compact subsets of $\mathbb{R}$.

5. $\mathcal{B}$ is not generated by the following:
(a) Any finite collection of subsets of $\mathbb{R}$
(b) The collection of real singletons.
(c) The collection of all finite subsets of $\mathbb{R}$.
(d) The collection of all amc subsets of $\mathbb{R}$.

6*. The representations (1)-(5) give equivalent formulations of $\bar{\mathcal{B}}$. [Let $\mathcal{C}$ denote the generating class for $\bar{\mathcal{B}}$ given in the definition, and let $\mathcal{D}$ denote the collection of sets that are in $\mathcal{B}$ or are $\mathcal{B}$-sets possibly augmented by one or both of $-\infty$ and $+\infty$. It is nearly automatic that $\mathcal{D}$ is a σ-field on $\bar{\mathbb{R}}$. Clearly an interval of the form $(a, b]$ where $-\infty \leq a < b \leq +\infty$ is in $\mathcal{D}$, so $\mathcal{C} \subseteq \mathcal{D}$, hence $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{D}) = \mathcal{D}$. The reverse inclusion $\mathcal{D} \subseteq \sigma(\mathcal{C})$ follows from the facts that (i) $\mathcal{D}$ contains $\mathcal{C}$ and (ii) $\mathcal{D}$ is clearly the smallest such σ-field containing $\mathcal{C}$. This gives $\sigma(\mathcal{C}) = \mathcal{D}$, and therefore the definition of $\bar{\mathcal{B}}$ is equivalent to the formulation given in (1). The equivalence between the definition and (2)-(5) follows in the same fashion.]

1.5 THE k-DIMENSIONAL BOREL σ-FIELD

This section generalizes the last section to k-dimensional Euclidean space for arbitrary $k \in \mathbb{N}$. There are some interesting questions that arise in higher dimensions; these cannot be asked when interest is restricted to one dimension.

Definition. Let $k \in \mathbb{N}$. The *k-dimensional Borel σ-field [on $\mathbb{R}^k$]* is denoted by $\mathcal{B}^k$, and is $\sigma(\{(\boldsymbol{a},\boldsymbol{b}] : \boldsymbol{a} < \boldsymbol{b}, \ \boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^k\})$. The sets in $\mathcal{B}^k$ are called *[k-dimensional] Borel sets.*

The symbol $\mathcal{B}^k$ is *not* to be construed as the k-fold Cartesian product $\mathcal{B} \times \cdots \times \mathcal{B}$ even though the notation suggests this; see the following discussion for further details.

Notation. We desire to adapt the notation of the previous section to $\mathbb{R}^k$ for any $k \in \mathbb{N}$. To that end, we will let $\mathcal{A}_1^k$ denote $\{(-\infty, \boldsymbol{x}] : \boldsymbol{x} \in \mathbb{R}^k\}$, $\mathcal{A}_1^{k*}$ will denote $\{(-\infty, \boldsymbol{x}] : \boldsymbol{x} \in \mathbb{Q}^k\}$, with $\mathcal{A}_2^k$, $\mathcal{A}_2^{k*}$, $\cdots$, $\mathcal{A}_8^k$, $\mathcal{A}_8^{k*}$ defined in analogous fashion; the classes $\mathcal{A}_9^k$ and $\mathcal{A}_{10}^k$ will denote the open subsets of $\mathbb{R}^k$ and the closed subsets of $\mathbb{R}^k$, respectively. When $k = 1$, these definitions reduce exactly to the definitions given in the last section.

According to the definition of $\mathcal{B}^k$, we have $\mathcal{B}^k = \sigma(\mathcal{A}_6^k)$. As in the previous section, various authors define $\mathcal{B}^k$ as the σ-field of subsets of $\mathbb{R}^k$ generated by $\mathcal{A}_1^{k*}$, $\mathcal{A}_7^{k*}$, and $\mathcal{A}_9^{k*}$, among other things. The following result says that any of the eighteen classes listed will generate $\mathcal{B}^k$.

Claim 1. $\mathcal{B}^k = \sigma(\mathcal{A}_1^k) = \cdots = \sigma(\mathcal{A}_{10}^k) = \sigma(\mathcal{A}_1^{k*}) = \cdots = \sigma(\mathcal{A}_8^{k*})$.

The proof of Claim 1 proceeds along the same tedious lines as does Claim 2 from the previous section; we merely need to make the natural modifications for k dimensions.

By Claim 1, the following types of subsets of $\mathbb{R}^k$ are k-dimensional Borel sets: all rectangles of all forms, all singletons, all finite subsets, all countably infinite sets, all open sets, all closed sets, and all sets that differ from a $\mathcal{B}^k$-set by at most countably many points. $\mathcal{B}^k$ will contain many more sets than these aforementioned types of sets, and, just as when $k = 1$, $\mathcal{B}^k$ will in the end be seen to include just about every set that one encounters in doing analysis on $\mathbb{R}^k$. Also, the questions (Q1)-(Q4) from the previous section, modified for k dimensions, have the same answers for all $k \geq 1$, with the proofs and careful discussions being postponed until later. In other words, we may state in colloquial terms that (i) there are $\mathfrak{c}$ sets in $\mathcal{B}^k$, (ii) there is no "nice" representation of a generic $\mathcal{B}^k$-set, (iii) there is a transfinite algorithm for obtaining all of $\mathcal{B}^k$ from countably infinite set operations performed on k-dimensional rectangles of all forms, and (iv) certain logical problems with the Axiom of Choice will arise in conjunction with future developments if we consider every subset of $\mathbb{R}^k$ as compared to merely some of them.

Notation. $\mathbb{R}^k$ is the k-fold Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$. For example, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and the elements of $\mathbb{R}^3$ are ordered triples (x_1, x_2, x_3), where $x_1, x_2, x_3 \in \mathbb{R}$. Strictly speaking, $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$, $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$, and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ are distinct; this is because the elements of $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ are ordered pairs of the form $((x_1, x_2), x_3)$, whereas the elements of $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$ are ordered pairs of the form $(x_1, (x_2, x_3))$. Clearly $((x_1, x_2), x_3) \neq (x_1, (x_2, x_3))$, and none of these equal (x_1, x_2, x_3). However, there is an obvious one-to-