Partial Differential Equations and the Finite Element Method

Pavel Šolín
The University of Texas at El Paso
Academy of Sciences of the Czech Republic

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PREFACE

Rien ne sert de courir,
il faut partir à point.
Jean de la Fontaine

Many physical processes in nature, whose correct understanding, prediction, and control are important to people, are described by equations that involve physical quantities together with their spatial and temporal rates of change *(partial derivatives)*. Among such processes are the weather, flow of liquids, deformation of solid bodies, heat transfer, chemical reactions, electromagnetics, and many others. Equations involving partial derivatives are called *partial differential equations (PDEs)*. The solutions to these equations are functions, as opposed to standard algebraic equations whose solutions are numbers. For most PDEs we are not able to find their exact solutions, and sometimes we do not even know whether a unique solution exists. For these reasons, in most cases the only way to solve PDEs arising in concrete engineering and scientific problems is to approximate their solutions numerically. Numerical methods for PDEs constitute an indivisible part of modern engineering and science.

The most general and efficient tool for the numerical solution of PDEs is the *Finite element method (FEM)*, which is based on the spatial subdivision of the physical domain into *finite elements* (often triangles or quadrilaterals in 2D and tetrahedra, bricks, or prisms in 3D), where the solution is approximated via a finite set of polynomial *shape functions*. In this way the original problem is transformed into a *discrete problem* for a finite number of unknown coefficients. It is worth mentioning that rather simple shape functions, such as affine or quadratic polynomials, have been used most frequently in the past due to their relatively low implementation cost. Nowadays, higher-order elements are becoming increasingly popular due to their excellent approximation properties and capability to reduce the size of finite element computations significantly.

The higher-order finite element methods, however, require a better knowledge of the underlying mathematics. In particular, the understanding of linear algebra and elementary functional analysis is necessary. In this book we follow the modern trend of building engineering finite element methods upon a solid mathematical foundation, which can be traced in several other recent finite element textbooks, as, e.g., [18] (membrane, beam and plate models), [29] (finite element analysis of shells), or [83] (edge elements for Maxwell's equations).

The contents at a glance

This book is aimed at graduate and Ph.D. students of all disciplines of computational engineering and science. It provides an introduction into the modern theory of partial differential equations, finite element methods, and their applications. The logical beginning of the text lies in Appendix A, which is a course in linear algebra and elementary functional analysis. This chapter is readable with minimum prerequisites and it contains many illustrative examples. Readers who trust their skills in function spaces and linear operators may skip Appendix A, but it will facilitate the study of PDEs and finite element methods to all others significantly.

The core Chapters 1–4 provide an introduction to the theory of PDEs and finite element methods. Chapter 5 is devoted to the numerical solution of ordinary differential equations (ODEs) which arise in the semidiscretization of time-dependent PDEs by the most frequently used *Method of lines (MOL)*. Emphasis is given to higher-order implicit one-step methods. Chapter 6 deals with Hermite and Argyris elements with application to fourth-order problems rooted in the bending of elastic beams and plates. Since the fourth-order problems are less standard than second-order equations, their physical background and derivation are discussed in more detail. Chapter 7 is a newcomer's introduction into computational electromagnetics. Explained are basic laws governing electromagnetics in both their integral and differential forms, material properties, constitutive relations, and interface conditions. Discussed are potentials and problems formulated in terms of potentials, and the time-domain and time-harmonic Maxwell's equations. The concept of Nédélec's *edge elements* for the Maxwell's equations is explained.

Appendix B deals with selected algorithmic and programming issues. We present a universal sparse matrix interface sMatrix which makes it possible to connect multiple sparse matrix solver packages simultaneously to a finite element solver. We mention the advantages of separating the finite element technology from the physics represented by concrete PDEs. Such approach is used in the implementation of a high-performance modular finite element system HERMES. This software is briefly described and applied to several challenging engineering problems formulated in terms of second-order elliptic PDEs and time-harmonic Maxwell's equations. Advantages of higher-order elements are demonstrated.

After studying this introductory text, the reader should be ready to read articles and monographs on advanced topics including a-posteriori error estimation and automatic adaptivity, mixed finite element formulations and saddle point problems, spectral finite element methods, finite element multigrid methods, hierarchic higher-order finite element methods (*hp*-FEM), and others (see, e.g., [9, 23, 69, 105] and [111]). Additional test and homework problems, along with an errata, will be maintained on my home page.

PAVEL ŠOLÍN

El Paso, Texas,
August, 2005

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I am deeply indebted to Prof. Ivo Doležel (Czech Technical University and Academy of Sciences of the Czech Republic), who is a theoretical electrical engineer with lively interest in computational mathematics, for providing me over the years with exciting practical problems to solve. Mainly thanks to him I learned to appreciate the engineer's point of view. The manuscript emerged from handouts, course notes, homeworks, and tests written for students. The students along with their interest and excitement were my main sources of motivation to write this book.

There is no way to express all my gratitude to my wife Dagmar for her support, understanding, and admirable patience during the two years of my work on the manuscript.

P. Š.

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CHAPTER 1

PARTIAL DIFFERENTIAL EQUATIONS

Many natural processes can be sufficiently well described on the macroscopic level, without taking into account the individual behavior of molecules, atoms, electrons, or other particles. The averaged quantities such as the deformation, density, velocity, pressure, temperature, concentration, or electromagnetic field are governed by partial differential equations (PDEs). These equations serve as a language for the formulation of many engineering and scientific problems. To give a few examples, PDEs are employed to predict and control the static and dynamic properties of constructions, flow of blood in human veins, flow of air past cars and airplanes, weather, thermal inhibition of tumors, heating and melting of metals, cleaning of air and water in urban facilities, burning of gas in vehicle engines, magnetic resonance imaging and computer tomography in medicine, and elsewhere. Most PDEs used in practice only contain the first and second partial derivatives (we call them second-order PDEs).

Chapter 1 provides an overview of basic facts and techniques that are essential for both the qualitative analysis and numerical solution of PDEs. After introducing the classification and mentioning some general properties of second-order equations in Section 1.1, we focus on specific properties of elliptic, parabolic, and hyperbolic PDEs in Sections 1.2–1.4. Indeed, there are important PDEs which are not of second order. To mention at least some of them, in Section 1.5 we discuss first-order hyperbolic problems that are frequently used to model transport processes such as, e.g., inviscid fluid flow. Fourth-order problems rooted in the bending of elastic beams and plates are discussed later in Chapter 6.

Partial Differential Equations and the Finite Element Method. By Pavel Šolín

1.1 SELECTED GENERAL PROPERTIES

Second-order PDEs (or PDE systems) encountered in physics usually are either elliptic, parabolic, or hyperbolic. Elliptic equations describe a special state of a physical system, which is characterized by the minimum of certain quantity (often energy). Parabolic problems in most cases describe the evolutionary process that leads to a steady state described by an elliptic equation. Hyperbolic equations describe the transport of some physical quantities or information, such as waves. Other types of second-order PDEs are said to be undetermined. In this introductory text we restrict ourselves to linear problems, since nonlinearities induce additional aspects whose understanding requires the knowledge of nonlinear functional analysis.

1.1.1 Classification and examples

Let $\mathcal{O}$ be an open connected set in $\mathbb{R}^n$. A sufficiently general form of a linear second-order PDE in n independent variables $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^T$ is

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial z_i}\left(a_{ij}\frac{\partial u}{\partial z_j}\right) + \sum_{i=1}^{n}\left(\frac{\partial}{\partial z_i}(b_i u) + c_i \frac{\partial u}{\partial z_i}\right) + a_0 u = f, \tag{1.1}$$

where $a_{ij} = a_{ij}(\boldsymbol{z}), b_i = b_i(\boldsymbol{z}), c_i = c_i(\boldsymbol{z}), a_0 = a_0(\boldsymbol{z})$ and $f = f(\boldsymbol{z})$. For all derivatives to exist in the classical sense, the solution and the coefficients have to satisfy the following regularity requirements: $u \in C^2(\mathcal{O}), a_{ij} \in C^1(\mathcal{O}), b_i \in C^1(\mathcal{O}), c_i \in C^1(\mathcal{O}), a_0 \in C(\mathcal{O})$, $f \in C(\mathcal{O})$. These regularity requirements will be reduced later when the PDE is formulated in the weak sense, and additional conditions will be imposed in order to ensure the existence and uniqueness of solution. If the functions a_{ij}, b_i, c_i, and a_0 are constants, the PDE is said to be with constant coefficients. Since the order of the partial derivatives can be switched for any twice continuously differentiable function u, it is possible to symmetrize the coefficients a_{ij} by defining

$$a_{ij}^{new} := (a_{ij}^{orig} + a_{ji}^{orig})/2$$

and adjusting the other coefficients accordingly so that the equation remains in the form (1.1). This is left to the reader as an exercise. Based on this observation, in the following we always will assume that the coefficient matrix $A(\boldsymbol{z}) = \{a_{ij}\}_{i,j=1}^{n}$ is symmetric.

Recall that a symmetric $n \times n$ matrix A is said to be positive definite if

$$\boldsymbol{v}^T A \boldsymbol{v} > 0 \qquad \text{for all } 0 \neq \boldsymbol{v} \in \mathbb{R}^n$$

and positive semidefinite if

$$\boldsymbol{v}^T A \boldsymbol{v} \geq 0 \qquad \text{for all } \boldsymbol{v} \in \mathbb{R}^n.$$

Analogously one defines negative definite and negative semidefinite matrices by turning the inequalities. Matrices which do not belong to any of these types are said to be indefinite.

Definition 1.1 (Elliptic, parabolic and hyperbolic equations) *Consider a second-order PDE of the form (1.1) with a symmetric coefficient matrix* $A(\boldsymbol{z}) = \{a_{ij}\}_{i,j=1}^{n}$.

1. *The equation is said to be* elliptic *at* $\boldsymbol{z} \in \mathcal{O}$ *if* $A(\boldsymbol{z})$ *is positive definite.*

2. *The equation is said to be* parabolic *at* $\boldsymbol{z} \in \mathcal{O}$ *if* $A(\boldsymbol{z})$ *is positive semidefinite, but not positive definite, and the rank of* $(A(\boldsymbol{z}), b(\boldsymbol{z}), c(\boldsymbol{z}))$ *is equal to* n.

3. *The equation is said to be* hyperbolic *at* $z \in \mathcal{O}$ *if* $A(z)$ *has one negative and* $n-1$ *positive eigenvalues.*

An equation is called elliptic, parabolic, *or* hyperbolic *in the set* $\mathcal{O}$ *if it is elliptic, parabolic, or hyperbolic everywhere in* $\mathcal{O}$*, respectively.*

Remark 1.1 (Temporal variable t**)** *In practice we distinguish between time-dependent and time-independent PDEs. If the equation is time-independent, we put* $n = d$ *and* $z = x$*, where* d *is the spatial dimension and* x *the spatial variable. This often is the case with elliptic equations. If the quantities in the equation depend on time, which often is the case with parabolic and hyperbolic equations, we put* $n = d+1$ *and* $z = (x, t)$*, where* t *is the temporal variable. In such case the set* $\mathcal{O}$ *represents some space-time domain. If the spatial part of the space-time domain* $\mathcal{O}$ *does not change in time, we talk about a space-time cylinder* $\Omega \times (0, T)$*, where* $\Omega \subset \mathbb{R}^d$ *and* $(0, T)$ *is the corresponding time interval.*

Notice that, strictly speaking, the type of the PDE in Definition 1.1 is not invariant under multiplication by -1. For example, the equation

$$-\Delta u = f \qquad \left(\text{where } \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \text{ in } \mathbb{R}^3\right) \tag{1.2}$$

is elliptic everywhere in $\mathbb{R}^3$ since its coefficient matrix $\boldsymbol{A}$ is positive definite,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

However, the type of the equation

$$\Delta u = -f$$

cannot be determined since its coefficient matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is negative definite. In such cases it is customary to multiply the equation by (-1) so that Definition 1.1 can be applied. Moreover, notice that Definition 1.1 only applies to second-order PDEs. Later in this text we will discuss two important cases outside of this classification: hyperbolic first-order systems in Section 1.5 and elliptic fourth-order problems in Chapter 6.

Remark 1.2 *Sometimes, linear second-order PDEs are found in a slightly different form*

$$-\sum_{i,j=1}^{n} \tilde{a}_{ij}(z) \frac{\partial^2 u}{\partial z_i \partial z_j} + \sum_{i=1}^{n} \tilde{b}_i(z) \frac{\partial u}{\partial z_i} + \tilde{a}_0(z) u = f(z), \tag{1.3}$$

usually with a symmetric coefficient matrix $\tilde{A}(z) = \{\tilde{a}_{ij}\}_{i,j=1}^{n}$*. When transforming (1.3) into the form (1.1), it is easy to see that the matrices* $\tilde{A}(z)$ *and* $A(z)$ *are identical, and* *thus either one can be used to determine the ellipticity, parabolicity, or hyperbolicity of the problem. Moreover, if the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$ are sufficiently smooth, the two forms are equivalent.*

Operator notation It is customary to write elliptic PDEs in a compact form

$$Lu = f,$$

where L defined by

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}(b_i u) + c_i\frac{\partial u}{\partial x_i}\right) + a_0 u \tag{1.4}$$

is a second-order elliptic differential operator. The part of L with the highest derivatives,

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right), \tag{1.5}$$

is called the principal (leading) part of L. Most parabolic and hyperbolic equations are motivated in physics, and therefore one of the independent variables usually is the time t. The typical operator form of parabolic equations is

$$\frac{\partial u}{\partial t} + Lu = f, \tag{1.6}$$

where L is an elliptic differential operator. Typical second-order hyperbolic equation can be seen in the form

$$\frac{\partial^2 u}{\partial t^2} + Lu = f, \tag{1.7}$$

where again L is an elliptic differential operator. The following examples show simple elliptic, parabolic, and hyperbolic equations.

■ **EXAMPLE 1.1 (Elliptic PDE: Potential equation of electrostatics)**

Let the function $\rho \in C(\overline{\Omega})$ represent the electric charge density in some open bounded set $\Omega \subset \mathbb{R}^d$. If the permittivity ϵ is constant in Ω, the distribution of the electric potential φ in Ω is governed by the Poisson equation

$$-\epsilon\Delta\varphi = \rho. \tag{1.8}$$

Notice that (1.8) does not possess a unique solution, since for any solution φ the function $\varphi + C$, where C is an arbitrary constant, also is a solution. In order to yield a well-posed problem, every elliptic equation has to be endowed with suitable boundary conditions. This will be discussed in Section 1.2.

■ **EXAMPLE 1.2 (Parabolic PDE: Heat transfer equation)**

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set and $q \in C(\overline{\Omega})$ the volume density of heat sources in Ω. If the thermal conductivity k, material density ϱ, and specific heat c are constant in Ω, the parabolic equation

$$\frac{\partial \theta}{\partial t} - \frac{k}{\varrho c}\Delta\theta = \frac{q}{\varrho c} \tag{1.9}$$

describes the evolution of the temperature $\theta(\boldsymbol{x}, t)$ in Ω. The steady state of the temperature ($\partial\theta/\partial t = 0$) is described by the corresponding elliptic equation

$$-k\Delta\theta = q.$$

Similarly to the previous case, the solution θ is not determined by (1.9) uniquely. Parabolic equations have to be endowed with both boundary and initial conditions in order to yield a well-posed problem. This will be discussed in Section 1.3.

■ **EXAMPLE 1.3 (Hyperbolic PDE: Wave equation)**

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. The speed of sound a can be considered constant in Ω if the motion of the air is sufficiently slow. Then the hyperbolic equation

$$\frac{\partial^2 p}{\partial t^2} - a^2\Delta p = 0 \tag{1.10}$$

describes the propagation of sound waves in Ω. Here the unknown function $p(\boldsymbol{x}, t)$ represents the pressure, or its fluctuations around some arbitrary constant equilibrium pressure. Again the function p is not determined by (1.10) uniquely. Hyperbolic equations have to be endowed with both boundary and initial conditions in order to yield a well-posed problem. Definition of boundary conditions for hyperbolic problems is more difficult compared to the elliptic or parabolic case, since generally they depend on the choice of the initial data and on the solution itself. We will return to this issue in Example 1.4 and in more detail in Section 1.5.

1.1.2 Hadamard's well-posedness

The notion of well-posedness of boundary-value problems for partial differential equations was established around 1932 by Jacques Salomon Hadamard.

J.S. Hadamard was a French mathematician who contributed significantly to the analysis of Taylor series and analytic functions of the complex variable, prime number theory, study of matrices and determinants, boundary value problems for partial differential equations, probability theory, Markov chains, several areas of mathematical physics, and education of mathematics.

Definition 1.2 (Hadamard's well-posedness) *A problem is said to be* well-posed *if*

1. *it has a unique solution,*
2. *the solution depends continuously on the given data.*

Otherwise the problem is ill-posed.

Figure 1.1 Jacques Salomon Hadamard (1865–1963).

As the reader may expect, well-posed problems are more pleasant to deal with than the ill-posed ones. The requirement of existence and uniqueness of solution is obvious. The other condition in Definition 1.2 denies well-posedness to problems with unstable solutions. From the point of view of numerical solution of PDEs, the computational domain Ω, boundary and initial conditions, and other parameters are not represented exactly in the computer model. Additional source of error is the finite computer arithmetics. If a problem is well-posed, one has a chance to compute a reasonable approximation of the unique exact solution as long as the data to the problem are approximated reasonably. Such expectation may not be realistic at all if the problem is ill-posed.

The concept of well-posedness deserves to be discussed in more detail. First let us show in Example 1.4 that well-posedness may be violated by endowing a PDE with wrong boundary conditions.

■ EXAMPLE 1.4 (Ill-posedness due to wrong boundary conditions)

Consider an interval $\Omega = (-a, a)$, $a > 0$, and the (inviscid) Burgers' equation

$$\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = 0. \tag{1.11}$$

This equation is endowed with the initial condition

$$u(x,0) = u_0(x) = x, \quad x \in \Omega, \tag{1.12}$$

where u_0 is a function continuous in $(-a, a)$ such that $u_0(\pm a) = \pm a$, and the boundary conditions

$$u(\pm a, t) = \pm a, \quad t > 0. \tag{1.13}$$

The (inviscid) Burgers' equation is an important representant of the class of first-order hyperbolic problems that will be studied in more detail in Section 1.5. In particular, after reading Paragraph 1.5.5 the reader will know that every function $u(x,t)$ that satisfies both equation (1.11) and initial condition (1.12) is constant along the lines

$$x_{x_0}(t) = x_0(t+1), \quad x_0 \in \Omega, \tag{1.14}$$

depicted in Figure 1.2.

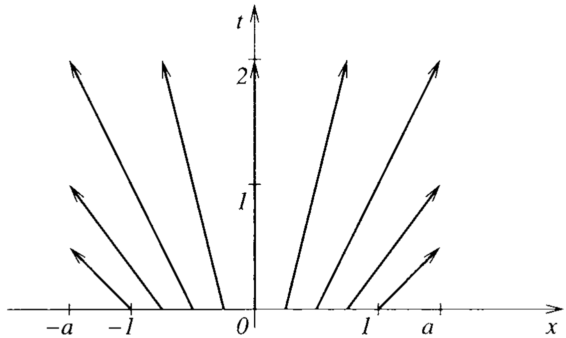

Figure 1.2 Isolines of the solution $u(x,t)$ of Burgers' equation.

It is easy to check the constantness of the solution u along the lines (1.14) by performing the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x_{x_0}(t), t).$$

From this fact it follows that the solution to (1.11), (1.12) cannot be constant in time at the endpoints of Ω. Hence the problem (1.11), (1.12), (1.13) has no solution.

Some problems are ill-posed because of their very nature, despite their initial and boundary conditions are defined appropriately. This is illustrated in Example 1.5.

■ EXAMPLE 1.5 (Ill-posed problem with unstable solution)

Consider the one-dimensional version of the heat transfer equation (1.9) with normalized coefficients,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.15}$$

describing the temperature distribution within a thin slab $\Omega = (0, \pi)$ in the time interval $(0, T)$. We choose an initial temperature distribution $u(x, 0) = u_0(x)$ such that $u_0(0) = u_0(\pi) = 0$, fix the temperature at the endpoints to $u(0) = u(\pi) = 0$ and ask about the solution $u(x,t)$ of (1.15) for $t \in (0, T)$. The initial condition $u_0(x)$ can be expressed by means of the Fourier expansion

$$u_0(x) = \sum_{n=1}^{\infty} c_n \sin(nx). \tag{1.16}$$

Thus it is easy to verify that the exact solution $u(x,t)$ has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx) \tag{1.17}$$

and hence that

$$u(x,T) = \sum_{n=1}^{\infty} c_n e^{-n^2 T} \sin(nx) \tag{1.18}$$

is the solution corresponding to the time $t = T$. Notice that the coefficients $c_n e^{-n^2 t}$ converge to zero very fast as the time grows, and therefore after a sufficiently long time T the solution will be very close to zero in Ω. Hence, the heat transfer problem evidently is a well-posed in the sense of Hadamard.

Now let us reverse the time by defining a new temporal variable $s = T - t$. The backward heat transfer equation has the form

$$\frac{\partial \hat{u}}{\partial s} + \frac{\partial^2 \hat{u}}{\partial x^2} = 0.$$

We consider an initial condition $\hat{u}_0(x)$ corresponding to $s = 0$, i.e., to $t = T$. Again, $\hat{u}_0(x)$ can be expressed as

$$\hat{u}_0(x) = \sum_{n=1}^{\infty} d_n \sin(nx), \tag{1.19}$$

and the exact solution $\hat{u}(x,s)$ has the form

$$\hat{u}(x,s) = \sum_{n=1}^{\infty} d_n e^{n^2 s} \sin(nx).$$

Notice that now the coefficients $d_n e^{n^2 s}$ are amplified exponentially as the backward temporal variable s grows. This means that the solution of the backward heat transfer equation does not depend continuously on the initial data $\hat{u}_0(x)$, i.e., that the backward problem is ill-posed.

Suppose that we calculate some numerical approximation of the solution $u(x,T)$ for some sufficiently large time T and then use it as the initial condition $\hat{u}_0(x)$ for the backward problem. What we will observe when solving the backward problem is that the solution $\hat{u}(x,s)$ begins to oscillate immediately and the computation ends with a floating point overflow or similar error very soon. Because of the ill-posedness of the backward problem, chances are slim that one can get close to the original initial condition $u_0(x)$ at $s = T$.

Remark 1.3 (Inverse problems) *The ill-posed backward heat transfer equation from Example 1.5 was an inverse problem. There are various types of ill-posed inverse problems: For example, it is an inverse problem to identify suitable initial state and/or parameters for some physical process to obtain a desired final state. Usually, the better-posed the forward problem, the worse the posedness of the inverse problem.*

1.1.3 General existence and uniqueness results

Prior to discussing various aspects of the elliptic, parabolic, and hyperbolic PDEs in Sections 1.2–1.5, we find it useful to mention a few important abstract existence and uniqueness results for general operator equations. Since this paragraph uses some abstract functional analysis, readers who find its contents too difficult may skip it in the first reading and continue with Section 1.2.

In the following we consider a pair of Hilbert spaces V and W, and an equation of the form

$$Lu = f, \tag{1.20}$$

where $L : D(L) \subset V \to W$ is a linear operator and $f \in W$. The existence of solution to (1.20) for any right-hand side $f \in W$ is equivalent to the condition $R(L) = W$, while the uniqueness of solution is equivalent to the condition $N(L) = \{0\}$.

Theorem 1.1 (Hahn–Banach) *Let U be a subspace of a (real or complex) normed space V, and $f \in U'$ a linear form over U. Then there exists an extension $g \in V'$ of f such that $g(u) = f(u)$ for all $u \in U$, moreover satisfying $\|g\|_{V'} = \|f\|_{U'}$.*

Proof: The proof can be found in standard functional-analytic textbooks. See, e.g., [34, 65] and [100]. ■

Theorem 1.1 has important consequences: If $v_0 \in V$ and $f(v_0) = 0$ for all $f \in V'$, then $v_0 = 0$. Further, for any $v_0 \in V$ there exists $f \in V'$ such that $\|f\|_V = 1$ and $f(v_0) = \|v_0\|_V$. The following result is used in the proof of the basic existence theorem: For any two disjoint subsets $A, B \subset V$, where A is compact and B convex, there exists $f \in V'$ and $\gamma \in \mathbb{R}$ such that $f(a) < \gamma < f(b)$ for all $a \in A$ and $b \in B$.

Theorem 1.2 (Basic existence result) *Let V, W be Hilbert spaces and $L : D(L) \subset V \to W$ a bounded linear operator. Then $R(L) = W$ if and only if both $R(L)$ is closed and $R(L)^{\perp} = \{0\}$.*

Proof: If $R(L) = W$, then obviously $R(L)$ is closed and $R(L)^{\perp} = \{0\}$. Conversely, assume that $R(L)$ is closed, $R(L)^{\perp} = \{0\}$ but $R(L) \neq W$. The linearity and boundedness of L implies that $R(L)$ is a closed subspace of W. Let $w \in W \setminus R(L)$. The set $\{w\}$ is compact and the closed set $R(L)$ obviously is convex. By the Hahn–Banach theorem there exists a $w^* \in W'$ such that $(w^*, w) > 0$ and $(w^*, Lv) = 0$ for all $v \in D(L)$. Therefore $0 \neq w^* \in R(L)^{\perp}$, which is a contradiction. ■

In order to see under what conditions $R(L)$ is closed, let us generalize the notion of continuity by introducing closed operators:

Definition 1.3 (Closed operator) *An operator $T : D(T) \subset V \to W$, where V and W are Banach spaces, is said to be* closed *if for any sequence $\{v_n\}_{n=1}^{\infty} \subset D(T)$, $v_n \to v$ and $T(v_n) \to w$ imply that $v \in D(T)$ and $w = Tv$.*

It is an easy exercise to show that every continuous operator is closed. However, there are closed operators which are not continuous:

■ **EXAMPLE 1.6 (Closed operator which is not continuous)**

Consider the interval $\Omega = (0, 1) \subset \mathbb{R}$, the Hilbert space $V = L^2(\Omega)$ and the Laplace operator $L : V \to V$, $Lu = -\Delta u = -u''$. This operator is not continuous, since, e.g., $Lv \notin V$ for $v = x^{-1/3} \in V$. We know that the space $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ (see Paragraph A.2.10). To show that L is closed in V, for an element $v \in V$ consider some sequence $\{v_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that $v_n \to v$, and such that the sequence $\{-\Delta v_n\}_{n=1}^\infty$ converges to some $w \in V$. Passing to the limit $n \to \infty$ in the relation

$$\int_\Omega -\Delta v_n \varphi \, \mathrm{d}\boldsymbol{x} = -\int_\Omega v_n \Delta\varphi \, \mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

we obtain

$$\int_\Omega w\varphi \, \mathrm{d}\boldsymbol{x} = -\int_\Omega v\Delta\varphi \, \mathrm{d}\boldsymbol{x} \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Therefore $w = -\Delta v$ and the operator L is closed.

Theorem 1.3 (Basic existence and uniqueness result) *Let V, W be Hilbert spaces and $L : D(L) \subset V \to W$ a closed linear operator. Assume that there exists a constant $C > 0$ such that*

$$\|Lv\|_W \geq C\|v\|_V \quad \textit{for all } v \in D(L) \tag{1.21}$$

(this inequality sometimes is called the stability or coercivity estimate). If $R(L)^\perp = \{0\}$, then the operator equation $Lu = f$ has a unique solution.

Proof: First let us verify that $R(L)$ is closed. Let $\{w_n\}_{n=1}^\infty \subset R(L)$ such that $w_n \to w$. Then there is a sequence $\{v_n\}_{n=1}^\infty \subset D(L)$ such that $w_n = Lv_n$. The stability estimate (1.21) implies that $C\|v_n - v_m\|_V \leq \|w_n - w_m\|_W$, which means that $\{v_n\}_{n=1}^\infty$ is a Cauchy sequence in V. Completeness of the Hilbert space V yields existence of a $v \in V$ such that $v_n \to v$. Since L is closed, we obtain $v \in D(L)$ and $w = Lv \in R(L)$. Theorem 1.2 yields the existence of a solution. The uniqueness of the solution follows immediately from the stability estimate (1.21). ∎

Now let us introduce the notion of monotonicity and show that strongly monotone linear operators satisfy the stability estimate (1.21):

Definition 1.4 (Monotonicity) *Let V be a Hilbert space and $L \in \mathcal{L}(V, V')$. The operator L is said to be* monotone *if*

$$\langle Lv, v\rangle \geq 0 \quad \textit{for all } v \in V, \tag{1.22}$$

it is strictly monotone *if*

$$\langle Lv, v\rangle > 0 \quad \textit{for all } 0 \neq v \in V, \tag{1.23}$$

and it is strongly monotone *if there exists a constant $C_L > 0$ such that*

$$\langle Lv, v\rangle \geq C_L\|v\|^2 \quad \textit{for all } v \in V. \tag{1.24}$$

For every $u \in V$ the element $Lu \in V'$ is a linear form. The symbol $\langle Lv, v\rangle$, which means the application of Lu to $v \in V$, is called duality pairing.