Wiley-Interscience Series in Discrete Mathematics and Optimization

INTRODUCTION TO DISCRETE DYNAMICAL SYSTEMS AND CHAOS

MARIO MARTELLI

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Introduction to Discrete Dynamical Systems and Chaos

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Introduction to Discrete Dynamical Systems and Chaos

MARIO MARTELLI
California State University
Fullerton

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To my wife and children

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PREFACE

The purpose of this book is to bring the fundamental ideas on discrete dynamical systems and chaos at the level of those undergraduates, usually in their junior year, who have completed the standard Calculus sequence, with the inclusion of functions of several variables and linear algebra. At this stage, students are in the best position for being exposed, during their college training, to the new ideas and developments generated in the last thirty years by the theory of discrete dynamical systems and chaos. The students' degree of sophistication permits the presentation of a broad range of topics and a fairly deep analysis of some nontrivial and historically interesting models. The importance and relevance of this exposure can hardly be described with better words than the ones used by R. Devaney (see [Devaney, 1989]). He writes: "*The field of dynamical systems and especially the study of chaotic systems has been hailed as one of the important breakthroughs in science in this century.*"

The book is divided into seven chapters and three appendices. Its content can be comfortably covered during a one-semester course, particularly if the teacher is satisfied with providing detailed proofs of only some fundamental results. As the title itself suggests, the topics of the book are limited to discrete dynamical systems. Several reasons have dictated this choice. The inclusion of both continuous and discrete systems would have created too large a body of material, with an inevitable loss of any in depth analysis. Moreover, a good understanding of continuous systems is hard to achieve without proper training in ordinary differential equations. Thus, their inclusion would have increased the prerequisites for the course. Another consideration that played an important role in the choice is the difficulty of establishing on theoretical grounds that a continuous system is chaotic. Chaos is one feature of dynamical systems that the book wants to present and analyze. It was considered awkward not to be in a position to prove that any continuous system is chaotic. A brief description of all chapters follows.

In Chapter 1 we present definitions and general ideas about discrete dynamical systems, together with some examples of significant interest derived from the recent research literature. In Section 1 we start with some examples of discrete dynamical systems and discuss the definition of discrete dynamical systems and the goals of the book. In Section 2 we introduce the standard definitions of fixed points, periodic orbits, and stability. In Section 3 we talk about limit points and aperiodic orbits, and we present a preliminary description of chaotic behavior. In Section 4 we give examples of systems, such as the system proposed by E.N. Lorenz to model atmospheric changes and the system proposed by J.J. Hopfield to model neural networks, which are later (Chapter 7) studied using the theory developed in the course.

Chapter 2 contains an extensive analysis of one-dimensional dynamical systems depending on one parameter. In the first section we introduce the cobweb method and the idea of conjugacy. In the second section we study the stability and instability of fixed points and periodic orbits. In Section 3 we present a result on global stability of fixed points. In Section 4 we introduce bifurcation, and we analyze this phenomenon both through examples and theoretically. The last section explores the implications of conjugacy and Li-Yorke chaos. The purpose of this chapter is to give the students some material to work on at the outset. Its only prerequisites are a few results from calculus of one variable, which are listed whenever needed, without proofs.

Chapter 3 contains an overview of those results of linear algebra and calculus of several variables which are likely to receive less attention during standard undergraduate courses. In the first section one finds something about the topology of $\mathbf{R}^q$ and of its structure as a normed vector space, the definition of continuity, and the equivalence of all norms. Section 2 deals with the operator norm of a matrix, the differentiability, first order-approximation, and the mean value inequality. The topics of this chapter are needed in Chapters 4 and 5.

In Chapter 4 we analyze discrete linear dynamical systems. Our study is based on three fundamental tools: the spectrum, a fundamental property of the spectral radius with respect to the operator norm of a linear map, and the spectral decomposition theorem. The first section explores the idea of representing the orbits of a linear system using eigenvectors. In Section 2 we study the case when the spectral radius is smaller than 1, and the case when all eigenvalues have modulus larger than 1. In Section 3 we present the spectral decomposition theorem, dividing the treatment into three cases: (1) when all eigenvalues are real and semisimple, (2) real but not semisimple, and (3) possibly complex. In Section 4 we investigate the saddle case, namely the case when some eigenvalues have modulus smaller than 1 and the others have modulus larger than 1. In Section 5 we analyze the case when at least one of the eigenvalues has modulus 1. Finally, in Section 6 we study affine systems, both in the case when 1 is not an eigenvalue and when it is.

With Chapter 5 we enter into the more challenging part of the book: the study of nonlinear systems in dimension higher than 1. In the first section we analyze systems having bounded invariant sets. Three types of maps are studied: contractive, dissipative, and quasi-bounded. We show here that the map proposed by Lorenz to model atmospheric behavior is dissipative and the one used by Hopfield to describe neural networks is quasi-bounded. Section 2 is devoted to maps having a unique fixed point that is a global attractor. Three classes of such maps are presented: contractions, triangular maps, and gradient maps. The third section deals with fixed points and periodic orbits that are sinks. In the fourth section we present repellers and saddles, with a brief excursion on stable and unstable manifolds. In Section 5 we discuss two fundamental results on bifurcation, including the Hopf case.

Chapter 6 is devoted to chaotic behavior. The first section opens the chapter with the definition of attractor and with a discussion of its relation to stability. In Section 2 we present a definition of chaotic dynamical systems based on the presence of a dense orbit and of its instability. Sensitivity with respect to initial conditions and other alternative definitions of chaos are also presented. In Section 3 we analyze the attractors of a chaotic system from the point of view of their dimension. Two types of dimension are discussed, the capacity and the correlation dimension. In the last section Lyapunov exponents are discussed together with their relation to stability and sensitivity with respect to initial conditions.

In Chapter 7 we present an extensive, although not complete analysis of the models introduced in Section 4 of Chapter 1, namely a blood-cell population model, predator-prey models for competition between two species, the model proposed by Lorenz as an approximation to the dynamics of atmospheric changes, and the Hopfield model of a neural network.

Appendix 1 contains an extensive collection of Mathematica programs that can be used to study discrete dynamical systems. They are referred to frequently throughout the book. The students and the teacher are free to use other symbolic manipulators such as Maple, or others. The ones presented here are simply for illustration purposes. An extensive analysis of a dynamical system is rarely possible without the significant help that can be provided by a computer, particularly when combined with at least a working knowledge of a powerful symbolic manipulator. It is hard, and perhaps even impossible, to get hands-on experience without using the enormous computing capability of a machine. Students are urged to learn how to use at least one of the many programs available. Some are designed strictly for the study of dynamical systems. Others address a much broader range of topics, and have quite a few features that can be exploited for a successful numerical study of discrete dynamical systems.

Appendix 2 has a list of references and possible team projects.

In Appendix 3 the reader finds short answers to selected problems. Many of the assigned problems can be solved without using a computing device. The answers to others are simpler to find with the aid of a symbolic manipulator. A few cannot be done without a computer, or a programmable calculator. A manual with detailed solutions of all problems is available from Wiley upon request.

The starred sections can be omitted without compromising the continuity of the presentation. They are clearly marked in the table of contents. The starred problems are easier to solve using the Mathematica programs listed in Appendix 1.

Wiley is providing a web site with some Mathematica programs which can be dowloaded and used in the investigation di dynamical systems, particularly in dimensions one and two. Please go to

ftp://ftp.wiley.com/public/sci_tech_med/dynamical_systems

Many scientists have written extensively about the importance of the topics presented in this book. It is hard to add something new to the things they have said so beautifully and appropriately. In particular, I believe that the number of mathematicians who feel that these topics should be made accessible to undergraduates largely exceeds the number of those who are still reluctant to "follow the trend." I belong to the first group, and this book is my attempt to provide one more tool for reaching the goal. Many friends have helped me in different ways. I am thankful to all of them. In particular I would like to mention Alfonso Albano, the late Stavros Busenberg, Courtney Coleman, Annalisa Crannell, Massimo Furi, and William Gearhart. When I was uncertain about the presentation of certain topics, or the most appropriate examples for clarifying definitions and theorems, they provided invaluable help. I am indebted to my students, to my daughter Monica, and to my son Teddy for finding numerous errors and misprints. I am thankful to my department, particularly to the chairman Dr. Jim Friel, for encouraging me in this enterprise by making a course on discrete dynamical systems and chaos mandatory for all math majors at CSUF.

Last, but not least, I would like to express my appreciation to the publisher for making the results of my effort available to the community of scientists and teachers. I sincerely hope that they will find this book useful.

Mario Martelli

Claremont, May 1999

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CHAPTER 1

DISCRETE DYNAMICAL SYSTEMS

SUMMARY

The chapter opens with some simple examples of discrete dynamical systems and continues with a formal definition of discrete dynamical systems and an outline of the goals of our study. Definitions of stationary state, periodic orbit, and the concept of stability are introduced in Section 2. The third section provides an informal description of chaotic behavior by means of aperiodic and unstable orbits. Some interesting examples of discrete dynamical systems derived from the current literature are presented in Section 4. A detailed analysis of these systems is provided in Chapter 7 using the theory developed in the previous chapters.

Section 1. DISCRETE DYNAMICAL SYSTEMS: DEFINITION

1. Examples of Discrete Dynamical Systems

We start our study of discrete dynamical systems with three simple examples.

Example 1.1.1 After searching the interest rates offered by several banks and savings and loans in our area, we have decided to invest \$5,000 with Everest Savings, which offers a 6.5% interest rate compounded monthly. The teller explains that at the end of every month the new principal P_{new} will be equal to the principal of the preceding month multiplied by 1+(.065/12). In other words,

$$P_{new} = \left(1 + \frac{.065}{12}\right)P_{old}.$$

Denoting by P_0 our original investment of \$5,000, we obtain that after 1,2,...,n months P_0 has grown to

$$P_1 = \left(1 + \frac{.065}{12}\right)P_0,\ P_2 = \left(1 + \frac{.065}{12}\right)P_1 = \left(1 + \frac{.065}{12}\right)^2 P_0,\ \ldots,$$

$$P_n = \left(1 + \frac{.065}{12}\right)P_{n-1} = \left(1 + \frac{.065}{12}\right)^n P_0.$$

The formula for P_n above can be generalized to every interest i and to every compounding period. For example, we learned that Mercury Savings is offering an interest rate of 6.8% compounded every 4 months. After m periods of 4 months, our investment has grown to

$$P_m = \left(1 + \frac{.068}{3}\right)P_{m-1} = \left(1 + \frac{.068}{3}\right)^m P_0.$$

Hence, 5 years with Everest Savings give a balance of $P_{60}=(1+.065/12)^{60}P_0=$ \$6914.09, while with Mercury Savings the balance is $P_{15}=(1+.068/3)^{15}P_0=$ \$6998.12. After 5 years we are slightly better off with Mercury Savings.

Let us use x_0, i, and m to indicate the initial investment, the interest and the number of compounding periods in a year, respectively. After n + 1 compounding periods, the amount x_{n+1} available to us is given by

$$x_{n+1} = (1 + i/m)x_n = (1 + i/m)^{n+1}x_0. \tag{1.1.1}$$

We have here a simple example of a discrete dynamical system. Let F(x) be the function

$$F(x) = (1 + i/m)x.$$

The system is governed by F and (1.1.1) can be rewritten in the form

$$x_{n+1} = F(x_n) = F^{n+1}(x_0), \tag{1.1.2}$$

where $F^{n+1}(x)$ represents the (n+1)th iterate of F. For example,

$$F^2(x) = F(F(x)) = (1 + i/m)F(x) = (1 + i/m)(1 + i/m)x = (1 + i/m)^2x.$$

In other words, $F^2(x)$ is evaluated by replacing x with $F(x)$ in the formula that defines F. To be more accurate we should write $F(i,m,x)=(1+i/m)x$, since both i and m play a role in the growth of the investment. However, we are interested mainly in the growth of x once i and m are fixed. Hence, we can still write $F(x)=(1+i/m)x$. The principal x is the **state variable** of the system. The interest i and the number m of compounding periods in a year are the **control parameters.**

The goals we are pursuing can be summarized as follows:

- Find the growth of any initial investment x_0, given i and m. In other words, look at the **evolution** of the state variable once the control parameters are **fixed**.
- Investigate how **changing** the control parameters (either i or m or both) affects the growth of the investment.

Frequently, both aspects are considered simultaneously.

We list some of the problems the reader may like to study:

- Given an interest i and a length m of the compounding periods, find how long it will take for an investment to double its original value.
- Find how much the interest should be for the investment to double in a certain period of time, assuming that the length of the compounding periods is known.
- Compare the combination of different interest rates and different compounding periods to see which bank offers the best deal.

Example 1.1.2 Our friend Ann, who is a biology major, is investigating the evolution of a colony of bacteria in the laboratory. In discussing the experiment we learn that it would be nice to have a formula that gives week by week the number N of bacteria per square inch. Ann tells us that in recent weeks there appears to be some kind of periodic behavior in the number of bacteria. One week their total appears to be higher than the number that can be supported by the laboratory environment (approximately 2.5×10^6 bacteria per square inch), and the week after the number appears to be lower.

To avoid dealing with very large numbers we let $x=N/10^6$ and we tell Ann that the proposed problem can be solved if we find the form of d_n and p_n in the discrete dynamical system:

$$x_{n+1} = x_n - d_n + p_n. \tag{1.1.3}$$

We explain that x_{n+1} represents the number of bacteria (divided by 10^6) at the beginning of the (n+1)th week of the experiment, x_n represents the same number the week before, and d_n, p_n are respectively the bacteria that died and were produced during the nth week. Ann tells us that experimental observations made so far suggest that about two-thirds of the bacteria die in any given week. Consequently, we suggest that $d_n=.7x_n$. Finding the form of p_n proves to be more challenging. She informs us that the growth of the colony was slow but steady at the beginning, and the periodic oscillations have been observed in the last few weeks. After some discussion and computer investigation we find that an acceptable form for p_n is $p_n=12x_n/(1+x_n{}^4)$. We rewrite (1.1.3) in the form

$$x_{n+1} = x_n - .7x_n + \frac{12x_n}{1 + x_n{}^4} = .3x_n + \frac{12x_n}{1 + x_n{}^4}. \qquad (1.1.4)$$

We explain to Ann that this is a discrete dynamical system governed by the function $F(x)=.3x+12x/(1+x^4)$, where x is the state variable of our system. Using F, we write (1.1.4) in the more compact form

$$x_{n+1} = F(x_n). \qquad (1.1.5)$$

We also remark that a more general version of (1.1.4) could be

$$x_{n+1} = x_n - ax_n + \frac{bx_n}{c + dx_n{}^p} = (1 - a)x_n + \frac{bx_n}{c + dx_n{}^p}, \qquad (1.1.6)$$

where a, b, c, d, and p are control parameters whose values can be adjusted to accommodate different experimental data. Then F becomes

$$F(a,b,c,d,p,x) = (1 - a)x + \frac{bx}{c + dx^p}, \quad \text{or} \quad F(\mathbf{a},x) = (1 - a)x + \frac{bx}{c + dx^p},$$

with $\mathbf{a}=(a,b,c,d,p)\in \mathbf{R}^5$ (since $\mathbf{a}$ has five components). Hence, (1.1.6) can be written in the form

$$x_{n+1} = F(\mathbf{a},x_n). \qquad (1.1.7)$$

However, since the evolution of the colony is usually studied with a, b, c, d, and p fixed, we may simply write $F(x)=(1-a)x+bx/(c+dx^p)$. Ann is very impressed by our analysis and is curious to see how our proposed model works. We go together to the computer lab, and open Mathematica. We use model (1.1.4). We ask her to suggest an initial value for the number of bacteria and she says that $x_0=1.5(\times 10^6)$ is a good estimate of the number with which the experiment started. Using a suitable Mathematica program (see Appendix 1, Section 2) and (1.1.4), we construct the first 60 states of the population of bacteria; namely, we compute $x_1=F(1.5)$, $x_2=F(x_1)$,..., $x_{60}=F(x_{59})$ and we plot the points $(0,x_0)$, $(1,x_1)$,..., $(60,x_{60})$. To make more evident how the population of bacteria is evolving, we connect all pairs $(i-1,x_{i-1})$, (i,x_i), $i=1,2,...,60$ with segments (see Fig. 1.1.1). The graph shows clearly that after an initial period of adjustment, the number of bacteria in the colony oscillates between one state above and one below 2.5×10^6, which, according to Ann, represents how many bacteria per square inch can be sustained by the laboratory environment.

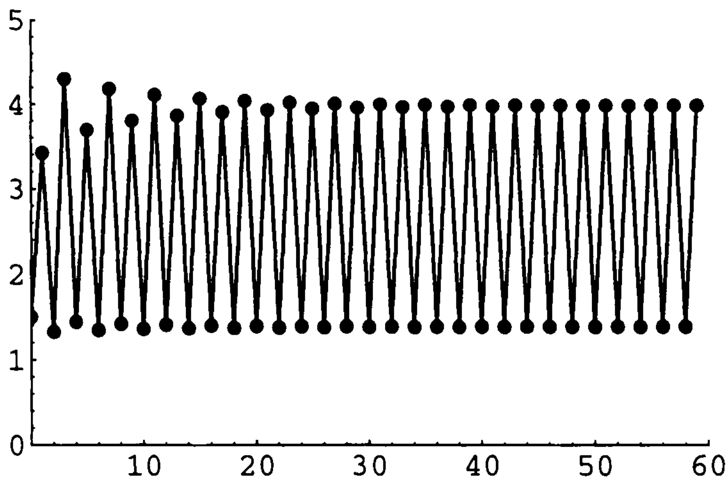

Fig. 1.1.1 It is evident that after an initial period of adjustment ("transient") the population of bacteria is oscillating periodically (with period 2).

We observe that the period of adjustment does not reflect the growth pattern observed by Ann at the beginning of the experiment. Different values of the parameters are needed to model this feature. For example, we could use a=.4, b=c=4, and p=2. We mention to Ann that the parameters cannot be expected to remain constant during an experiment. We also point out that a different choice of a, b, c, d, and p in (1.1.6) can model a behavior of the colony more complex than the steady growth or the periodic oscillations. For example, leaving a, b, c, and d as in (1.1.4) and selecting p=5 gives the system

$$x_{n+1} = .3x_n + \frac{12x_n}{1+x_n^5} \cdot \qquad (1.1.8)$$

With the same initial condition x_0=1.5, we obtain a graph which suggests that the number of bacteria in the colony is evolving in an erratic manner (see Fig. 1.1.2).

Ann now believes that mathematics can be very useful in biology! She is curious to know if there is a model in which the three patterns of steady growth, periodic oscillations, and erratic evolution can be produced by changing **only** the death rate a. We tell her that the question is very interesting and we promise to investigate it.

Example 1.1.3 After a few days, Ann returns with another problem. Producing some newspaper clippings she points out that the ash whitefly (*Siphoninus phillyreae*) was introduced in Southern California around August 18, 1988 and multiplied so quickly that 48 of the 56 California counties were literally invaded by these pests.

About three years later the entomologists from UC Riverside imported, mass reared and released the ash whitefly's natural enemy, a tiny black wasp (*Encarsia inaron*). The summer infestation density of the ash whitefly before the mass release of *E. inaron* averaged 8 to 21 whiteflies/cm^2 of infested leaf. Within two years of *E. inaron* release the infestation density dropped to .32 to 2.18 whiteflies/cm^2 leaf (see [Pickett et al., 1996]). Ann would like to know if we can design a mathematical model of this situation.

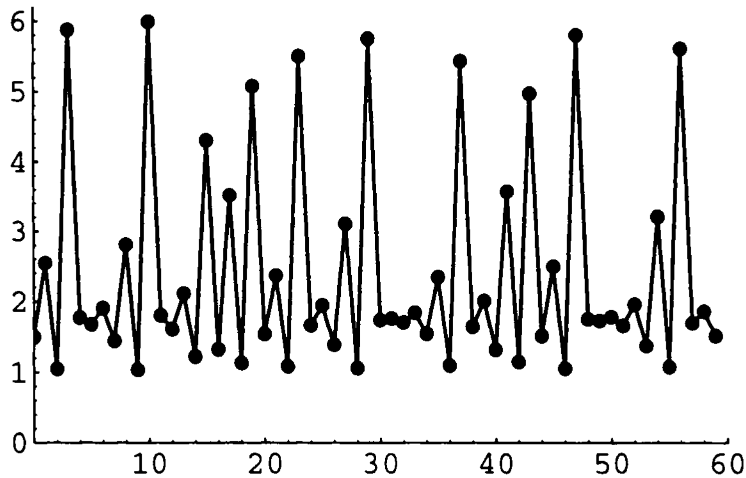

Fig. 1.1.2 It appears that when the evolution of the colony of bacteria is governed by (1.1.8) the number of bacteria follows an erratic pattern.

After giving the problem some serious consideration, we feel that the populations of whiteflies and tiny wasps should undergo periodic one-year oscillations, with possible different levels of density for both populations during the different seasons. At the outset we are inclined to adopt a model for the whiteflies similar to the one proposed before for the colony of bacteria, possibly with different constants and with an extra term to account for the whiteflies destroyed by the *E. inaron*. However, after some research in the library, we realize that this dynamical process belongs to the class of predator-prey problems, which have been studied extensively since 1920 (see Section 4 of this chapter for details). Guided by the literature in this area, and after some computer experiments we choose the form

$$\begin{cases} x_{n+1} = 1.01x_n - .008x_ny_n \\ y_{n+1} = .9y_n + .04y_n(1.01x_n - .065x_ny_n). \end{cases} \tag{1.1.9}$$

We show (1.1.9) to Ann and we explain that the state variable x represents the whiteflies (x=2 means an average of 2 whiteflies/cm^2 of infested leaf) and the state variable y the wasps (y=3 means an average of 3 black wasps/cm^2 of whiteflies infested leaf). We point out that in the absence of wasps, the population of whiteflies grows exponentially (see Fig. 1.1.3), since

$$x_1 = 1.01x_0, x_2 = 1.01x_1 = (1.01)^2x_0, \ldots, x_n = (1.01)^nx_0, \ldots$$

and $(1.01)^n \to \infty$ as $n \to \infty$. This is not very realistic in the long run, but it was certainly true for the first few months after the infestation began. In the absence of whiteflies the wasps become extinct since

$$y_1 = .9y_0,\ y_2 = .9y_1 = (.9)^2 y_0, \ldots,\ y_n = (.9)^n y_0, \ldots$$

and $(.9)^n \to 0$ as $n \to \infty$. The function governing the system is

$$\mathbf{F(x)} = F(x,y) = (1.01x-.008xy,\ .9y+.04xy(1.01-.065y)). \qquad (1.1.10)$$

F is a function from $\mathbf{R}^2$ into $\mathbf{R}^2$, although for obvious reasons we are interested only in the first quadrant of $\mathbf{R}^2$, namely $x \geq 0$ and $y \geq 0$. The component functions of F are

$$F_1(\mathbf{x}) = F_1(x,y) = 1.01x - .008xy,\ F_2(\mathbf{x}) = F_2(x,y) = .9y + .04xy(1.01 - .065y).$$

System (1.1.9) can be rewritten in the form

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n). \qquad (1.1.11)$$

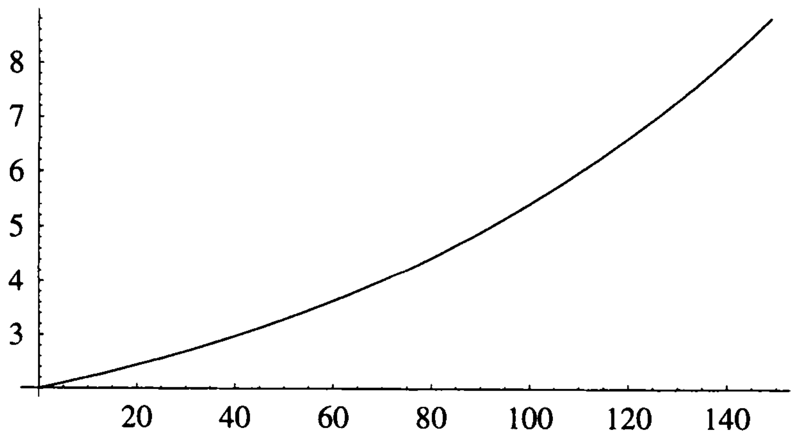

Fig. 1.1.3 The number of whiteflies/cm^2 of infested leaf grows exponentially in the absence of a natural enemy.

We assume that in agreement with the available data, the average number of whiteflies/cm^2 of infested leaf was $x_0=14$ at the mass release of black wasps, and let $y_0=2$. Hence, the initial state is $\mathbf{x}_0=(14,2)$. We compute the first 400 states of the populations of both species and plot in one graph (i,x_i), and in another (i,y_i) for $i=0,1,\ldots,400$. We superimpose the two graphs (Mathematica uses the Show command: see Section 2 of Appendix 1) to obtain Fig. 1.1.4, in which the larger dots represent the whiteflies. We see that their population decreases dramatically. We do not see in the graph the convergence of both populations to the steady state (2.15,1.25), which appears to be a reasonable long-term outcome and is in good agreement with the data provided.

Ann is truly amazed by the proposed model, but we feel that it is not perfect and that further adjustments are needed. For example, we can make the model more general by introducing some parameters, as we did in the case of the colony of bacteria. A starting point could be

$$\mathrm{F}(\mathbf{x}) = \mathrm{F}(x,y) = (\mathrm{a}x - .008xy,\ \mathrm{c}y + .04xy(\mathrm{a} - .065y)). \tag{1.1.12}$$

In (1.1.12) a and c are two control parameters, a>1 and c∈(0,1), which can be adjusted to fit experimental data and to better represent the behavior of the two populations. Then, to be more accurate, we should write

$$\mathrm{F}(\mathbf{a},\mathbf{x}) = (\mathrm{a}x - .008xy,\ \mathrm{c}y + .04xy(\mathrm{a} - .065y)) \tag{1.1.13}$$

where $\mathbf{x}=(x,y)$ is the vector in $\mathbf{R}^2$ which represents the two state variables of the system (number of whiteflies and number of black wasps) and $\mathbf{a}=(\mathrm{a,c})$ is a vector in $\mathbf{R}^2$ which represents the (positive) control parameters of the system. The dynamical system becomes

$$\mathbf{x}_{n+1} = \mathrm{F}(\mathbf{a},\mathbf{x}_n). \tag{1.1.14}$$

However, when parameters a and c are regarded as fixed numbers, we may prefer the symbol F(**x**) to F(**a**,**x**) and write (1.1.14) in the simpler form

$$\mathbf{x}_{n+1} = \mathrm{F}(\mathbf{x}_n). \tag{1.1.15}$$

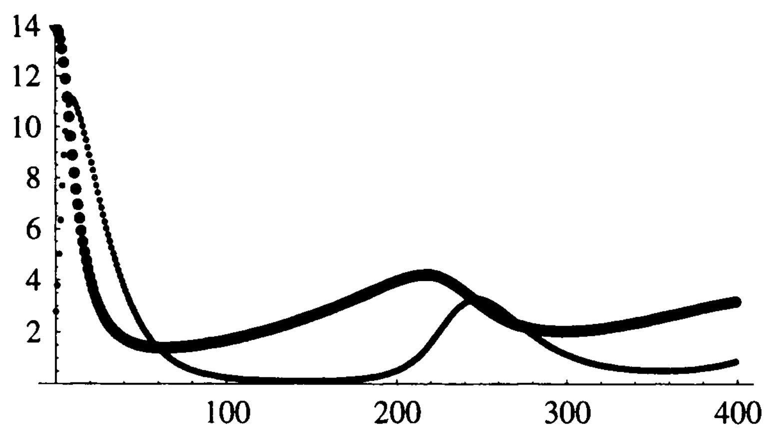

Fig. 1.1.4 The population of whiteflies (larger dots) decreases dramatically after the introduction of black wasps. The two populations converge to the state (2.15, 1.25), in agreement with the information provided.

Problems

Throughout the book, problems with a * have questions whose numerical and/or graphical solution may require a computer or a calculator. The Mathematica programs that can be used in the numerical investigation of these problems are contained in Appendix 1. When the problem requires you to find the time or find the fixed point or similar quantities, you should find the exact value whenever possible.

1. * An investor makes a deposit of \$50,000 at 7% compounded monthly. Find the time needed for the principal to double, namely to reach \$100,000.

2. * An investor makes a deposit of \$5,000 at 6% compounded monthly. After 30 months the investor adds \$3,000. At that time the bank changes the compounding period to a trimester. Find the balance after 45 months.

3. * A colony of bacteria starts with $x_0=2.4(\times 10^5)$ bacteria/inch^2. It is assumed that the growth of the colony is governed by $F(x)=.6x+4x^6e^{-1.5x}$ where $x(\times 10^5)$ represents the number of bacteria/inch^2. The time interval between the nth and the (n+1)th generation is one week. Write the dynamical system that governs the evolution of the colony and find out how many bacteria/inch^2 there are after 10 weeks (for a suitable Mathematica program to find x_{10}, see Section 2 of Appendix 1).

4. * Examine the same situation of Problem 3 assuming that x_0 is the same but that $F(x)=.4x+4x^6e^{-1.5x}$.

5. * Find those values of x such that $x=.3x+12x/(1+x^4)$ (see Example 1.1.2). These are the stationary states (or fixed points) of the system (you can use the programs of Appendix 1, Section 4).

6. * Assume that the growth of a colony of bacteria is governed by $F(x)=.6x+4x^6e^{-1.5x}$. Find those values of x such that $x=F(x)$ (see Appendix 1, Section 4). Generalize F with the introduction of control parameters.

7. In Example 1.1.3 [see (1.1.10)] there are two different state vectors $\mathbf{x}$ in the first quadrant such that $\mathbf{x}=F(\mathbf{x})$. One of them is $\mathbf{x}=\mathbf{0}$. Find the other one.

8. * Explore the consequences of changing the values of parameters a and c in the system of Example 1.1.3 [see equality (1.1.12)]. Introduce other parameters and investigate how the evolution of the system is affected by changes in these new parameters.

9. * Use the Orbit program (Appendix 1, Section 2) to illustrate the growth of the investment in Problem 1. Compare the outcomes with Everest Savings and Mercury Savings. Make some comments.

10. * Explore the possibility of solving the problem proposed by Ann in connection with Example 1.1.2.

2. Definition of Discrete Dynamical Systems

After looking at the previous examples, we see that a discrete dynamical system is a relation of the form

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n). \tag{1.1.16}$$

The function F(**x**) of (1.1.16) may contain a number of control parameters **a**= (a,b,c,...). In this case the function should be written F(**a**,**x**). However, as mentioned before, when the parameters are considered fixed, we may prefer the notation F(**x**) to F(**a**,**x**). How to arrive at (1.1.16), its meaning, use, and limitations are described briefly below. We assume that a given real dynamical process has to be studied, as in the three examples presented before.

• We identify the state variables of the system. For example, in the biological process involving the two competing species of whiteflies and black wasps, we are interested in knowing how each species affects the evolution of the other. Hence, two state variables are needed. We used the coordinates of the two-dimensional state vector $\mathbf{x}=(x,y)$ to represent the average number of whiteflies and black wasps per infested leaf. In general, the state vector **x** is q-dimensional, i.e., $\mathbf{x}\in \mathbf{R}^q$.

• We identify possible control parameters of the system. In this step we look for those parameters that affect the evolution of the state variables. In the first example we found two: the interest rate i and the number m of compounding periods per year. Hence the parameter vector **a** had two components, $\mathbf{a}=(i,m)$, $\mathbf{a}\in \mathbf{R}^2$. In the second example we proposed five parameters: a,b,c,d, and p. Hence, **a** had five components, $\mathbf{a}=(a,b,c,d,p)$, $\mathbf{a}\in \mathbf{R}^5$. In general, **a** has m components, i.e., $\mathbf{a}\in \mathbf{R}^m$, with m and q usually different.

• The next step is to determine the mathematical relations that translate the laws governing the evolution of the process. As suggested by (1.1.16) and as we have seen in the three examples, these relations are embodied by a function F which depends on the state vector **x** and on the parameter vector **a**, although we have agreed to consider **a** constant. The range of F is in the same space where the state vector **x** lives. In fact, in the first two examples the state variable was simply x, and F was a function from **R** into **R**, although in both cases we were interested only in $x\geq 0$. In the second example the state variables were two: x and y. Thus we considered the vector $\mathbf{x}=(x,y)$ and F was a function from $\mathbf{R}^2$ into $\mathbf{R}^2$, although once more we were interested only in $x\geq 0$ and $y\geq 0$.

In Examples 1.1.2 and 1.1.3 we have seen that determining the exact form of F might be a nontrivial task. The situation analyzed in Example 1.1.3 was particularly challenging and we may still have doubts that our model is the best for the problem proposed. In general, F is defined on some subsets $U\subset\mathbf{R}^m$ and $V\subset\mathbf{R}^q$ with range in $\mathbf{R}^q$. We can write $F : U\times V\to\mathbf{R}^q$, where $U\times V$ is the standard Cartesian product of the two sets U and V. Notice that F is identified by its q component functions $F_i : U\times V\to\mathbf{R}$. For example, in (1.1.13) we have $F_1(\mathbf{a},\mathbf{x})=ax-.008xy$, $F_2(\mathbf{a},\mathbf{x}) =cy+.04xy(a-.065y)$.

Determining the proper form of the function F is a central part of the modeling process. It is not an easy task. As a rule of thumb, one can say that there is an optimal range for the level of complexity of any model. Below this range, the model is biased toward the theoretical side, and above this range the model loses its synthetic advantage. Large areas of uncertainty remain between these two extremes, and it may be difficult, or even impossible, to design an "optimal model," namely, a model in which all advantages and key features are incorporated.

• We are now ready to write the dynamical system (1.1.16), which tells us how the (n+1)th state of the vector $\mathbf{x}$, denoted by $\mathbf{x}_{n+1}$, is obtained once the nth state $\mathbf{x}_n$ is known. The 0-state, $\mathbf{x}_0$, is called the **initial state.** It can be any vector $\mathbf{x}$ in the domain of F. The evolution of the system starting from $\mathbf{x}_0$ is given by the sequence

$$\mathbf{x}_0, \mathbf{x}_1 = F(\mathbf{x}_0), \mathbf{x}_2 = F(\mathbf{x}_1) = F(F(\mathbf{x}_0)), \mathbf{x}_3 = F(\mathbf{x}_2) = F(F(F(\mathbf{x}_0))), \dots .$$

We usually write $F^2(\mathbf{x})$, $F^3(\mathbf{x})$, ... in place of $F(F(\mathbf{x}))$, $F(F(F(\mathbf{x})))$, ... Hence, we have

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n) = F^{n+1}(\mathbf{x}_0). \qquad (1.1.17)$$

Equality (1.1.17) will be used repeatedly throughout the book.

Definition 1.1.1
The sequence $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n, \dots\}$ is denoted by $O(\mathbf{x}_0)$ and is called the **orbit** or **trajectory** of the system starting from $\mathbf{x}_0$.

As remarked previously, we shall always assume that as an orbit $O(\mathbf{x}_0)$ evolves, the control parameters are kept constant, although we realize that this limitation may be unrealistic in real processes (see Example 1.1.2).

• The time interval between two successive states of an orbit is usually suggested by the real process itself. For example, $\mathbf{x}_{n+1}$ could be separated from $\mathbf{x}_n$ by one hour, one day, one week, one month, etc. In the examples presented before, the time interval was the compounding period (in Example 1.1.1) and one week (in Example 1.1.2). No time interval was provided for Example 1.1.3.

The reader is probably thinking that in real processes the time variable evolves in a continuous rather than a discrete manner. Our approach does not follow this continuous evolution. We **sample** the state of the system at fixed time intervals. This strategy can be better understood if the reader thinks about the standard practice of measuring body temperature every 6 to 8 hours, even though the temperature undergoes continuous changes. With proper choice of the time step, the technique provides very useful information on the behavior of the real process. The time intervals are always assumed to be equally spaced and are denoted by 0, 1, ..., n, ...

The following additional examples should make the reader more familiar and comfortable with the topics discussed so far.

Example 1.1.4 Assume that the value x_{n+1} of the state variable x at time n+1 depends on the value x_n at time n according to the relation

$$x_{n+1} = ax_n(1 - x_n).$$

The function that governs the system is $F(x)=ax(1-x)$, and q=m=1. Once more, we should really consider F as depending on the variable x and on the parameter a, namely $F(a,x)=ax(1-x)$. This dynamical system is called **logistic**. For modeling reasons it is normally required that $x \in [0,1]$ and $a \in [0,4]$. Hence $U=[0,4]$ and $V=[0,1]$. The time step is left open.

Given an initial state x_0, we have

$$x_1 = F(x_0) = ax_0(1 - x_0),\ x_2 = F(x_1) = ax_1(1 - x_1).$$

Since $x_1=ax_0(1-x_0)$, we obtain

$$x_2 = a(ax_0(1 - x_0))(1 - ax_0(1 - x_0)) = a^2x_0(1 - x_0) - a^3x_0{}^2(1 - x_0)^2 = F^2(x_0).$$

Hence, x_2 is computed by replacing x_0 with $F(x_0)$ in the definition of F. Similarly, x_3, x_4, ... are obtained by replacing x_0 with $F^2(x_0)$, $F^3(x_0)$,... in the definition of F (for the numerical aspect of this process, see Appendix 1, Section 2, Subsection 1).

Example 1.1.5 Assume that in a two-species population the size of each species at time n+1 depends on the size at time n according to the relations

$$\begin{cases} x_{n+1} = ax_n(1 - x_n - y_n) \\ y_{n+1} = bx_n y_n. \end{cases}$$

Then $\mathbf{x}=(x,y)\in \mathbf{R}^2$ and $\mathbf{a}=(a,b)\in \mathbf{R}^2$. The function governing the system is

$$F(\mathbf{x}) = F(x,y) = (ax(1 - x - y),\ bxy), \qquad (1.1.18)$$

with component functions $F_1(\mathbf{x})=ax(1-x-y)$ and $F_2(\mathbf{x})=bxy$.

Technically, we should regard F as a function of $\mathbf{x}=(x,y)$ and $\mathbf{a}=(a,b)$ and write

$$F(\mathbf{a},\mathbf{x}) = F((a,b),\ (x,y)) = (F_1(\mathbf{a},\mathbf{x}),\ F_2(\mathbf{a},\mathbf{x})),$$

where $F_1(\mathbf{a},\mathbf{x})=ax(1-x-y)$ and $F_2(\mathbf{a},\mathbf{x})=bxy$. However, we can still use form (1.1.18) when we study the evolution of the two species. In fact, in this case, the parameters a and b are considered constant, although their exact value may not be specified.

The state variable and the parameters are assumed to be positive. Hence, $U=\{(a,b): a\geq 0,\ b\geq 0\}$, $V=\{(x,y): x\geq 0,\ y\geq 0\}$. The time interval between two successive states is left open.

Goals of this book

In this book we **do not study** the strategies and methods used to derive (1.1.16) from a real dynamical process. This topic belongs to a book on mathematical modeling and is not addressed here. Also, we do not normally investigate and discuss the length of the time step between successive states. The starting point of our study are relations of the form (1.1.16). We plan to achieve the following distinct but strictly related goals:

1. Analyze the behavior of (1.1.16) for different values of $\mathbf{x}_0$, considering the parameters fixed.

2. Study the changes in this behavior as the parameters are changed.

We provide now a more detailed description of the two goals and illustrate, with a simple example, how they can be achieved.

• The first task is to determine the behavior of the orbits of (1.1.16) (we also say "the orbits of F") for different choices of the initial condition $\mathbf{x}_0$. We focus on the long-term behavior of the orbits. In other words, we look at what an orbit does for n very large, or more rigorously, as $n \to \infty$. We do not intend to neglect the "transient behavior," namely, what is happening at the beginning of an orbit, i.e., for small values of n. In fact, frequently we will be looking at an entire orbit $O(\mathbf{x}_0)$. Our main interest, however, is to determine the fate of $\mathbf{x}_n$ as $n \to \infty$. We call this part state space analysis of the dynamical system.

• The second task is to study the changes in the characteristics and behavior of (1.1.16) brought to bear by changes in the parameters. We call this part parameter space analysis of the dynamical system. During this stage of our investigation we should use, for obvious reasons, $F(\mathbf{a},\mathbf{x})$ rather than $F(\mathbf{x})$.

When **a** is a scalar (denoted by a, $a \in I$, $I \subset \mathbf{R}$, I an interval) the set of functions $\{F(a,\mathbf{x}) : a \in I\}$ is called a **one-parameter** family of maps. We shall pay particular attention to the properties of such families.

The following simple example illustrates how both tasks can be accomplished.

Example 1.1.6 Let $F(x)=ax+2$. This is a one-parameter family of maps and we can write $\{F(a,x)=ax+2, a \in \mathbf{R}\}$. To illustrate the two types of investigations mentioned above, first consider the parameter a fixed, for example, a=3/4. Hence, $F(x)=(3/4)x+2$. Notice that if we select $x_0=8$ we have $x_1=(3/4)8+2=8$. Thus, $x_1=x_0$. We call this value of x a **fixed point** of F since 8=F(8), and we denote it by x_s.

Let us study the orbits of the system $x_{n+1}=(3/4)x_n+2$. Using the fixed point 8 we can write

$$x_{n+1} = (3/4)(x_n - 8) + 8. \qquad (1.1.19)$$

Given x_0, we have

$$x_1 = (3/4)(x_0 - 8) + 8,\ x_2 = (3/4)(x_1 - 8) + 8 = (3/4)^2(x_0 - 8) + 8, \ldots .$$

In the expression for x_2 we have replaced x_1 with $(3/4)(x_0-8)+8$. After n+1 steps we arrive at

$$x_{n+1} = (3/4)^{n+1}(x_0 - 8) + 8. \qquad (1.1.20)$$

Since $(3/4)^{n+1} \to 0$ as $n \to \infty$ we obtain that $x_{n+1} \to 8$ as $n \to \infty$. In other words, every orbit converges to the fixed point $x_s=8$ no matter what the initial condition $x_0 \in \mathbf{R}$ might be.

Let us investigate what changes the system goes through as the parameter a is changed. First, the fixed point is now the solution of $x=ax+2$. We see that there is no fixed point when a=1, and there is only one, $x_s(a)=2/(1-a)$, when $a\neq 1$. We have written $x_s(a)$ since the fixed point "changes" with a. When $a\neq 1$ we can use (1.1.19) with 3/4 replaced by a and 8 replaced by $x_s(a)$. By writing x_s instead of $x_s(a)$, the relation (1.1.19) assumes the form

$$x_{n+1} = a(x_n - x_s) + x_s, \qquad (1.1.21)$$

and (1.1.20) becomes

$$x_{n+1} = a^{n+1}(x_0 - x_s) + x_s. \qquad (1.1.22)$$

With $|a|<1$ every orbit converges to $x_s=2/(1-a)$ since $a^{n+1}\to 0$ as $n\to\infty$. With $|a|>1$ the orbit of every initial state $x_0\neq x_s$ goes to infinity. For a=−1 the fixed point is $x_s=1$, and given $x_0\neq 1$, we have $x_1=-x_0+2$, $x_2=-x_1+2=-(-x_0+2)+2=x_0$, $x_3=-x_2+2=-x_0+2=x_1$. Hence the orbit is x_0, $x_1=-x_0+2$, x_0, $-x_0+2$, x_0, ... We say that $O(x_0)$ is periodic of period 2. For a=1 we have $x_1=x_0+2$, $x_2=x_1+2=x_0+4$, $x_3=x_2+2=x_0+8$,... . Every orbit goes to $+\infty$.

We have completed the parameter space analysis of our system by specifying how the system $x_{n+1}=ax_n+2$ behaves for all possible values of the parameter $a\in\mathbf{R}$.

Example 1.1.6 gives a little taste of how the two goals can be achieved. It also shows that the two tasks, although different, are strictly related. In many cases we will not make an effort to identify which task we are pursuing.

Sometimes (1.1.16) is replaced by slightly more general forms, like

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n, \mathbf{x}_{n-1}). \qquad (1.1.23)$$

Equation (1.1.23) tells us that the state of the orbit at time n+1 depends directly from its states at time n and n−1. We call (1.1.23) a discrete dynamical system with a delay of a one-time unit. We can eliminate the delay by increasing the dimension of the system. We set $\mathbf{y}_{n+1}=\mathbf{x}_n$ and rewrite (1.1.23) as follows:

$$\begin{cases} \mathbf{x}_{n+1} = F(\mathbf{x}_n, \mathbf{y}_n) \\ \mathbf{y}_{n+1} = \mathbf{x}_n. \end{cases} \qquad (1.1.24)$$

By setting $G(\mathbf{x}, \mathbf{y})=(F(\mathbf{x},\mathbf{y}),\mathbf{x})$ and $\mathbf{z}=(\mathbf{x},\mathbf{y})$ we arrive at

$$\mathbf{z}_{n+1} = G(\mathbf{z}_n), \qquad (1.1.25)$$

which is of the same form as (1.1.16). More complicated cases are also possible, and they are solved analogously, as the following examples show.

Example 1.1.7 Let $x_{n+1}=ax_{n-1}(1-x_n)$. This one-dimensional dynamical system, containing a delay of a one-time unit, can be replaced by a two-dimensional system with no delay by setting $y_{n+1}=x_n$. We obtain the system

$$\begin{cases} x_{n+1} = ay_n(1-x_n) \\ y_{n+1} = x_n. \end{cases}$$

In vector form we get $\mathbf{x}_{n+1}=G(\mathbf{x}_n)$ where $\mathbf{x}=(x,y)$ and $G(\mathbf{x})=(ay(1-x),x)$.

Example 1.1.8 Let $x_{n+1}=ax_n(1-x_{n-2})$. The dynamical system contains a delay of two-time units. We can replace it with a three-dimensional system with no delay. Set $y_{n+1}=x_n$ and $z_{n+1}=y_n$. We obtain

$$\begin{cases} x_{n+1} = ax_n(1-z_n) \\ y_{n+1} = x_n \\ z_{n+1} = y_n. \end{cases}$$

In vector form we have $\mathbf{x}_{n+1}=G(\mathbf{x}_n)$, $\mathbf{x}=(x,y,z)$, $G(\mathbf{x})=(ax(1-z),x,y)$.

Example 1.1.9 Assume that $\mathbf{x}_{n+1}=F(\mathbf{x}_n,\mathbf{x}_{n-1},\mathbf{x}_{n-2})$. Let $\mathbf{y}_{n+1}=\mathbf{x}_n$ and $\mathbf{w}_{n+1}=\mathbf{y}_n$. We obtain the system

$$\begin{cases} \mathbf{x}_{n+1} = F(\mathbf{x}_n, \mathbf{y}_n, \mathbf{w}_n) \\ \mathbf{y}_{n+1} = \mathbf{x}_n \\ \mathbf{w}_{n+1} = \mathbf{y}_n. \end{cases}$$

Setting $\mathbf{z}=(\mathbf{x},\mathbf{y},\mathbf{w})$, $G(\mathbf{x},\mathbf{y},\mathbf{w})=(F(\mathbf{x},\mathbf{y},\mathbf{w}),\mathbf{x},\mathbf{y})$, we can write $\mathbf{z}_{n+1}=G(\mathbf{z}_n)$ which is of the form (1.1.16).

In this book we always assume that $\mathbf{x}_{n+1}$ depends directly **only on** $\mathbf{x}_n$. Systems with delay, i.e., when $\mathbf{x}_{n+1}$ depends directly on one or more states of the form $\mathbf{x}_{n-k}$, $k\geq 1$, are replaced by higher-dimensional systems with no delay.

Problems

1. Let $F(\mathbf{a},x)=x(ax-b)$. The dynamical system governed by F is $x_{n+1}=x_n(ax_n-b)$. Identify the state variables and the control parameters.

2. Let $\mathbf{x}_{n+1}=F(\mathbf{a},\mathbf{x}_n)$, where $F(\mathbf{a},\mathbf{x})=(ax(1-x-y),bxy)$. Write the dynamical system explicitly (namely, $x_{n+1}=\ldots,y_{n+1}=\ldots$), pointing out its state variables, its parameters, and its component functions.

3. Let $F(\mathbf{a},\mathbf{x})=(a(x-y),bx-y-xz,xy-cz)$. Write explicitly the discrete dynamical system governed by F. Identify the state variables, the control parameters, and the component functions.

4. Let $F(a,x)=-ax^2+1$. Write explicitly $F(a,F(a,x))$, usually denoted by $F^2(a,x)$ (you can use the programs of Appendix 1, Section 2).

5. Let $F(\mathbf{a},x)=x(a-bx)$. Write explicitly $F(\mathbf{a},F(\mathbf{a},x))=F^2(\mathbf{a},x)$.

6. Let $F(a,x) =-ax^2+1$. What is the degree in x of the polynomial $F^3(a,x)$? What about the degree in a? Generalize to F^n, the nth iterate of F.

7. Let $x_{n+1}=x_n+x_{n-1}$. This is the Fibonacci sequence and it can be regarded as a one-dimensional discrete dynamical system with a delay of one-time unit. Replace it with a two-dimensional system with no delay.

8. Let $x_{n+1}=ax_{n-1}(1-x_{n-2})$. Replace this one-dimensional system with a three-dimensional system with no delay.

9. Let

$$\begin{cases} x_{n+1} &= 2x_n - .2x_{n-1}y_{n-1} \\ y_{n+1} &= y_n + .1x_{n-1}y_{n-1}. \end{cases}$$

Replace the two-dimensional system having a delay of a one-time unit with a four-dimensional system with no delay.

10. Let $\mathbf{x}_{n+1}=F(\mathbf{x}_n)$ with F given by (1.1.12). Assume that at certain point in time the y species becomes extinct. Denote by x_0 the size of x at that time. Verify that x will grow exponentially from that point on (recall that $a>1$). Is this unrestricted growth reasonable?

11. Let $\mathbf{x}_{n+1}=F(\mathbf{x}_n)$ with F given by (1.1.12). Assume that at a certain point in time the x species becomes extinct. Denote by y_0 the size of y at that time. Verify that y will become extinct (recall that $0<c<1$). Is this outcome reasonable?

Section 2. STATIONARY STATES AND PERIODIC ORBITS

1. Stationary States

Definition 1.2.1
A point $\mathbf{x}_0$ is called a **stationary state** of (1.1.16) if

$$\mathbf{x}_1 = F(\mathbf{x}_0) = \mathbf{x}_0. \tag{1.2.1}$$

The symbol $\mathbf{x}_s$ will be used to denote a stationary state. Each $\mathbf{x}_s$ can be regarded either as a state of the dynamical system $\mathbf{x}_{n+1}=F(\mathbf{x}_n)$ satisfying (1.2.1) or as a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x}=F(\mathbf{x})$. For this reason we shall also call $\mathbf{x}_s$ a **fixed point** of F.

Example 1.2.1 Every stationary state of the system $x_{n+1}=ax_n(1-x_n)$ must satisfy the equation $x=ax(1-x)$. We see that $x_s=0$ is a stationary state regardless of the value of a. Assuming that $x\neq 0$, we obtain

$$1 = a(1 - x) \quad \text{or} \quad ax = a - 1. \tag{1.2.2}$$

Therefore, a second stationary state is $x_s(a)=(a-1)/a$. We have written $x_s(a)$ since the point changes with a. For every a we can visualize the fixed points of $F(x)=ax(1-x)$ since they are given by the intersection of the graph of F with the line $y=x$ (see Fig. 1.2.1 for a=3).

Example 1.2.2 The stationary states of $(x_{n+1},y_{n+1})=(ax_n-bx_ny_n,cy_n+dx_ny_n)$ are the solutions of the system

$$\begin{cases} x = ax - bxy \\ y = cy + dxy. \end{cases}$$

The point (0,0) is a solution regardless of the values of a,b,c, and d. Assume that $0<c<1$ and $a>1$. It follows that $x=0$ if and only if $y=0$. Thus, additional stationary states satisfy the inequality $xy\neq 0$. Dividing the first equation by x and the second by y, we obtain the system

$$\begin{cases} 1 = a - by \\ 1 = c + dx. \end{cases} \tag{1.2.3}$$

Hence, $x = (1-c)/d$, $y =(a-1)/b$ or $\mathbf{x}_s(\mathbf{a})=((1-c)/d,(a-1)/b)$.

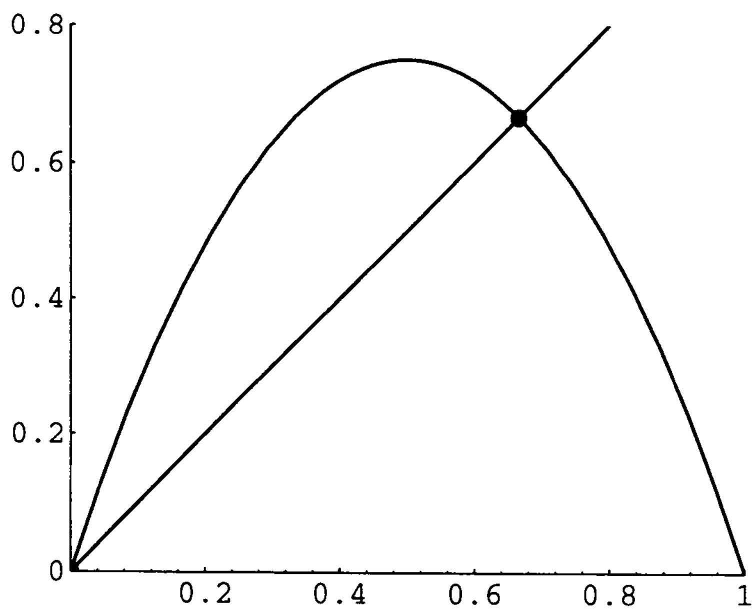

Fig. 1.2.1 The fixed points of the dynamical system governed by the function $F(x)=3x(1-x)$ are the intersections of the graph of F with the line $y=x$.

Notice that any orbit starting from a stationary state $\mathbf{x}_s$ will not move away from it. This situation is possible in theory, but not in practice, since every process undergoes small fluctuations which are normally not accounted for in a model. These fluctuations will force the system out of $\mathbf{x}_s$. This observation, together with others which will be discussed later, motivates the introduction of the idea of **stability**. Some fundamental definitions are presented in this chapter. Other concepts related to stability will be discussed in later chapters.

Stable stationary states

Definition 1.2.2
A stationary state $\mathbf{x}_s$ is **stable** if for every r>0 there exists δ>0 such that

$$\|\mathbf{x}_0 - \mathbf{x}_s\| \le \delta \text{ implies that } \|\mathbf{x}_n - \mathbf{x}_s\| \le r \text{ for all } n \ge 1. \tag{1.2.4}$$

In other words, once we have chosen how close we want to remain to $\mathbf{x}_s$ in the future (choice of r), we can find how close we must start at the beginning (existence of δ).

The symbol $\|\mathbf{x}\|$ denotes the "Euclidean norm" of **x**, defined by

$$\|\mathbf{x}\| = \|(x_1, ..., x_q)\| = (x_1^2 + ... + x_q^2)^{.5}. \tag{1.2.5}$$

In **R** the Euclidean norm is the usual absolute value.

Example 1.2.3 Let $F(x)=1-x$. The point $x_s=.5$ is the only fixed point of F. For every other initial state x_0 we have $x_1=F(x_0)=1-x_0$, $F(F(x_0))=F(1-x_0)=x_0$. Thus, $|x_n-.5|=|x_0-.5|$ for all n=1,2,... . The fixed point $x_s=.5$ is stable (see Fig. 1.2.2). In the definition of stability the number δ can be selected equal to r.

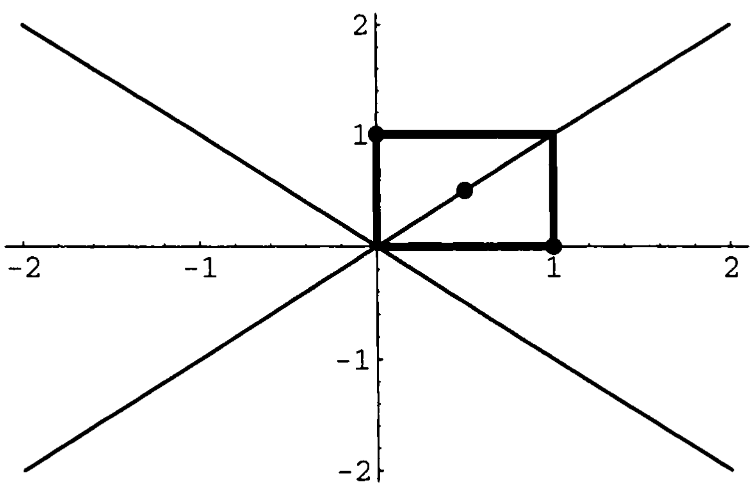

Fig. 1.2.2 This graph shows the fixed point .5 and graphically illustrates its stability (see Example 1.2.3).