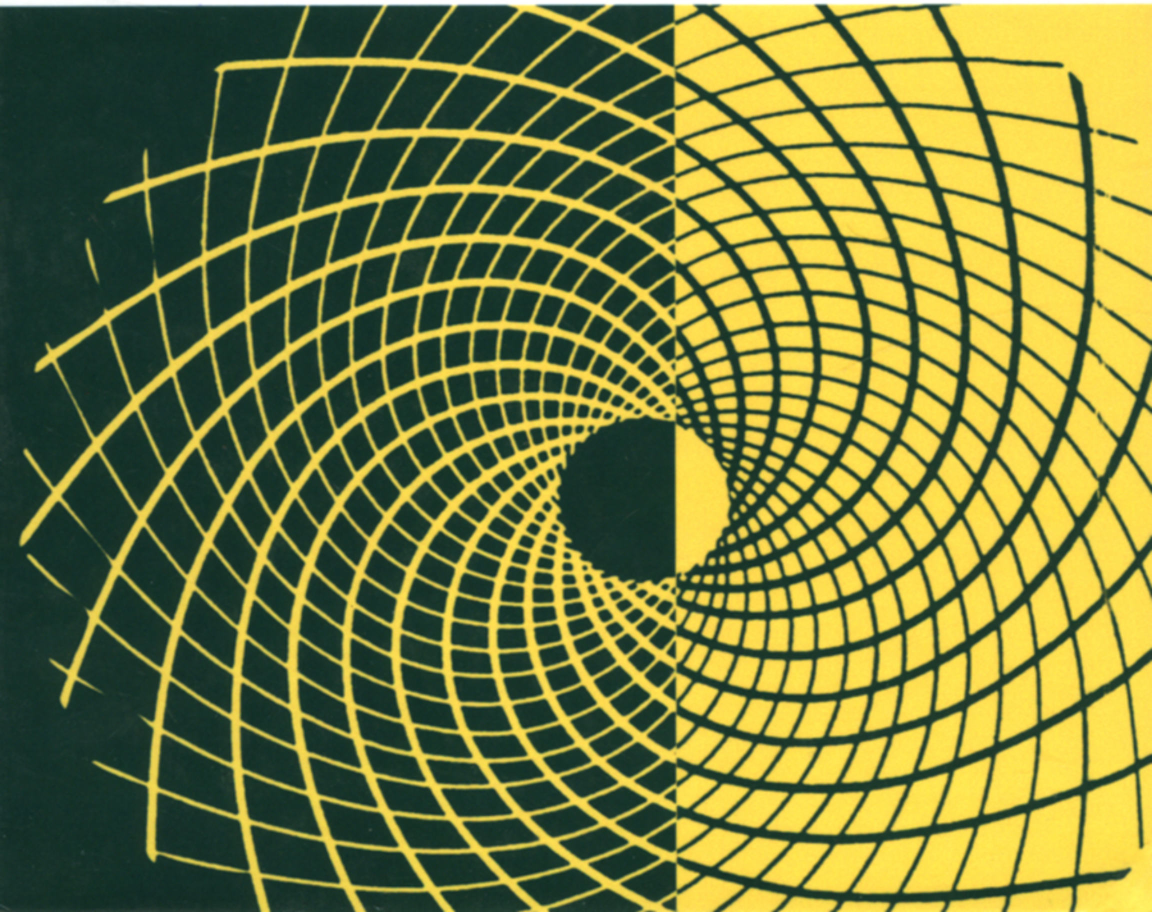
Pure and Applied Mathematics
A Wiley-Interscience Series of Texts, Monographs, and Tracts

Fibonacci and Lucas Numbers with Applications

Thomas Koshy



This page intentionally left blank

FIBONACCI AND LUCAS NUMBERS WITH APPLICATIONS

PURE AND APPLIED MATHEMATICS

A Wiley-Interscience Series of Texts, Monographs, and Tracts

Founded by RICHARD COURANT
Editors: MYRON B. ALLEN III, DAVID A. COX, PETER LAX
Editors Emeriti: PETER HILTON, HARRY HOCHSTADT, JOHN TOLAND

A complete list of the titles in this series appears at the end of this volume.

FIBONACCI AND LUCAS NUMBERS WITH APPLICATIONS

THOMAS KOSHY
Framingham State College

A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York · Chichester · Weinheim · Brisbane · Singapore · Toronto

This book is printed on acid-free paper. ♾

Copyright © 2001 by John Wiley & Sons, Inc. All rights reserved.

Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as 107 or 108 of the 1976 United States Copyright Act, without either the
prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4744. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158-0012, (212) 850-6011, fax (212) 850-6008, E-Mail: PERMREQ@WILEY.COM.

For ordering and customer service, call 1-800-CALL WILEY.

Library of Congress Cataloging-in-Publication Data:
Koshy, Thomas.
Fibonacci and Lucas numbers with applications/Thomas Koshy.
p. cm. — (Pure and applied mathematics: a Wiley-Interscience series of texts, monographs, and tracts)
"A Wiley-Interscience publication."
Includes bibliographical references and index.
ISBN 0-471-39969-8 (cloth : alk. paper)
1. Fibonacci numbers. 2. Lucas numbers. I. Title II. Pure and applied mathematics (John Wiley & Sons: Unnumbered)

QA246.5.K67 2001
512'.72—dc21 2001017506

10 9 8 7 6 5 4 3 2 1

To

Suresh, Neethu, and Sheeba

This page intentionally left blank

CONTENTS

This page intentionally left blank

PREFACE

> Man has the faculty of becoming completely absorbed in one subject, no matter how trivial and no subject is so trivial that it will not assume infinite proportions if one's entire attention is devoted to it.
>
> —Tolstoy, *War and Peace*

The Twin Shining Stars

The Fibonacci sequence and the Lucas sequence are the two shining stars in the vast array of integer sequences. They have fascinated both amateurs and professional mathematicians for centuries, and they continue to charm us with their beauty, their abundant applications, and their ubiquitous habit of occurring in totally surprising and unrelated places. They continue to be a fertile ground for creative amateurs and mathematicians alike.

This book grew out of my fascination with the intriguing beauty and rich applications of the twin sequences. It has been my long-cherished dream to study and to assemble the myriad properties of both Fibonacci and Lucas numbers, developed over the centuries, and to catalog their applications to various disciplines in an orderly and enjoyable fashion.

An enormous amount of information is available in the mathematical literature on Fibonacci and Lucas numbers; but, unfortunately, most of it is widely scattered in numerous journals, so it is not easily accessible to many, especially to nonprofessionals. In this book, I have collected and presented materials from a wide range of sources, so that the finished volume represents, to the best of my knowledge, the largest comprehensive study of this area to date.

Although many Fibonacci enthusiasts know the basics of Fibonacci and Lucas numbers, there are a multitude of discoveries about properties and applications that may be less familiar. Fibonacci and Lucas numbers are also a source of great fun; teachers and professors often use them to generate excitement among students, who find that the sequences stimulate their intellectual curiosity and sharpen their mathematical skills, such as pattern recognition, conjecturing, proof techniques, and problem-solving.

Audience

This book is intended for a wide audience. College undergraduate and graduate students often opt to study Fibonacci and Lucas numbers because they find them challenging and exciting. Often many students propose new and interesting problems in periodicals. It is certainly delightful that students often pursue Fibonacci and Lucas numbers for their senior and master's theses.

High school students have enjoyed exploring this material for a number of years. Using Fibonacci and Lucas topics, students at Framingham High School in Massachusetts, for instance, have published many of their Fibonacci and Lucas discoveries in *Mathematics Teacher*.

I have also included a large amount of advanced material to challenge mathematically sophisticated enthusiasts and professionals in such diverse fields as art, biology, chemistry, electrical engineering, neurophysiology, physics, and music. It is my hope that this book will serve them as a valuable resource in exploring new applications and discoveries, and advance the frontiers of mathematical knowledge.

Organization

In the interest of manageability, the book is divided into forty-seven short chapters. Most conclude with numeric and theoretical exercises for Fibonacci enthusiasts to explore, conjecture, and confirm. I hope that the exercises are as exciting for readers as they are for me. Where the omission can be made without sacrificing the essence of development or focus, I have omitted some of the long, tedious proofs of theorems. The solutions to all odd-numbered exercises are given in the back of the book.

Salient Features

Salient features of this book include: a user-friendly, historical approach; a nonintimidating style; a wealth of identities, applications, and exercises of varying degrees of difficulty and sophistication; links to graph theory, matrices, geometry, and trigonometry; the stock market; and relationships to geometry and information from everyday life. For example, works of art are discussed vis-à-vis the *Golden Ratio*, one of the most intriguing irrational numbers.

Interdisciplinary Appeal

The book contains numerous and fascinating applications to a wide spectrum of disciplines and endeavors; These include art, architecture, biology, chemistry, chess, electrical engineering, geometry, graph theory, music, origami, poetry, physics, physiology, psychology, neurophysiology, sewage/water treatment, snow plowing, stock market trading, and trigonometry. Most of the applications are well within the reach of mathematically sophisticated amateurs, although they vary in difficulty and sophistication.

Historical Perspective

Throughout, I have tried to present historical background for the material, and to humanize the discourse by giving the name and affiliation of every contributor to the field, as well as the year of contribution. My apologies to any discoverers whose names or affiliations are missing; I would be pleased to hear of any such inadvertent omissions.

Puzzles

The book contains several numeric puzzles based on Fibonacci numbers. In addition, it contains several popular geometric paradoxes, again rooted in Fibonacci numbers, which are certainly a source of excitement and surprise.

List of Symbols

A glossary of symbols follows this preface. Readers can find a list of the fundamental properties from the theory of numbers and the theory of matrices in the Appendix. Those who are curious about their proofs will find them in my forthcoming book on number theory.

I would be delighted to hear from Fibonacci enthusiasts about any possible inadvertent errors. If any reader should have questions, or should discover any additional properties and applications, I would be more than happy to hear about them.

Acknowledgments

I am pleased to take this opportunity to thank a number of people who have helped to improve the manuscript with their constructive suggestions, comments, and support.

I am deeply indebted to the following reviewers for their boundless enthusiasm and input:

N. Gauthier	Royal Military College, Kingston, Canada
H. W. Gould	West Virginia University, Morgantown, West Virginia
Marjorie Johnson	Santa Clara, California
Thomas E. Moore	Bridgewater State College, Bridgewater, Massachusetts
Carl Pomerance	University of Georgia, Athens, Georgia
M. N. S. Swamy	Concordia University, Montreal, Canada
Monte J. Zerger	Adams State College, Alamosa, Colorado

Thanks also to Angelo DiDomenico of Framingham High School, who read an early version of the work and offered valuable suggestions; to Margarite Roumas for her superb editorial assistance; to Kevin Jackson-Mead for preparing the canonical prime factorizations of the Lucas numbers L_{51} through L_{100} and for proofreading the entire work; and to Madelyn Good at the Framingham State College Library, who tracked down copies of many articles cited in the references, and many books on loan from other institutions; and to the staff at Wiley, especially, Steve Quigley and Heather Haselkorn for their cooperation, support, and encouragement, and their confidence in the project.

THOMAS KOSHY
tkoshy@frc.mass.edu

Framingham, Massachusetts
June 2001

LIST OF SYMBOLS

SYMBOL	MEANING
$\mathbb{N}$	the set of positive integers $1, 2, 3, 4, \ldots$
$\mathbf{W}$	the set of whole numbers $0, 1, 2, 3, \ldots$
$\mathbb{R}$	the set of real numbers
$\{s_n\}_1^\infty = \{s_n\}$	sequence with general term s_n
$\sum_{i=k}^{i=m} a_i = \sum_{i=k}^{m} a_i = \sum_{k}^{m} a_i$	$a_k + a_{k+1} + \cdots + a_m$
$\sum_{i \in I} a_i$	sum of the values of a_i as i runs over the various values in I
$\sum_P a_i$	sum of the values of a_i, where i satisfies certain properties P
$\sum_i \sum_j a_{ij}$	$\sum_i (\sum_j a_{ij})$
$\prod_{i=k}^{i=m} a_i = \prod_{i=k}^{m} a_i = \prod_{k}^{m} a_i$	$a_k a_{k+1} \cdots a_m$
$n!$ (n factorial)	$n(n-1) \cdots 3 \cdot 2 \cdot 1$, where $0! = 1$
$\|x\|$	the absolute value of x
$\lfloor x \rfloor$ (the floor of x)	the greatest integer $\leq x$
$\lceil x \rceil$ (the ceiling of x)	the least integer $\geq x$
PMI	the principle of mathematical induction
a div b	the quotient when a is divided by b
a mod b	the remainder when a is divided by b
$a \mid b$	a is a factor of b
$a \nmid b$	a is not a factor of b
$\{x, y, z\}$	the set consisting of the elements x, y, and z

$\{x \mid P(x)\}$	the set of elements with property $P(x)$
$\lvert A \rvert$	the number of elements in set A
$A \cup B$	the union of sets A and B
$A \cap B$	the intersection of sets A and B
(a, b)	the greatest common factor of a and b
$[a, b]$	the least common factor of a and b
$A = (a_{ij})_{m \times n}$	$m \times n$ matrix A whose ijth element is a_{ij}
$\lvert A \rvert$	the determinant of matrix A
$\in$	belongs to
$\approx$	is approximately equal to
$\equiv$	is congruent to
$\therefore$	therefore
∞	the infinity symbol
$\blacksquare$	the end of a proof, solution, or an example
$\overline{AB}$	the line segment with endpoints A and B
AB	the length of the line segment $\overline{AB}$
$\overleftrightarrow{AB}$	the line containing the points A and B
$\overrightarrow{AB}$	the ray AB
$\angle ABC$	the angle ABC
$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel
$\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$	the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular
RHS	right-hand side
LHS	the left-hand side

FIBONACCI AND LUCAS NUMBERS WITH APPLICATIONS

This page intentionally left blank

1

LEONARDO FIBONACCI

Leonardo Fibonacci, also called Leonardo Pisano or Leonard of Pisa, was the most outstanding mathematician of the European Middle Ages. Little is known about his life except for the few facts he gives in his mathematical writings. Ironically, none of his contemporaries mention him in any document that survives.

Fibonacci (Fig. 1.1) was born around 1170 into the Bonacci family of Pisa, a prosperous mercantile center. ("Fibonacci" is a contraction of "Filius Bonacci," son of Bonacci.) His father Guglielmo (William) was a successful merchant, who wanted his son to follow his trade.

Around 1190, when Guglielmo was appointed collector of customs in the Algerian city of Bugia (now Bougie), he brought Leonardo there to learn the art of computation. In Bougie, Fibonacci received his early education from a Muslim schoolmaster, who introduced him to the Indo-Arabic numeration system and Indo-Arabic computational techniques. He also introduced Fibonacci to a book on algebra, *Hisâb al-jabr w'al-muqabâlah*, written by the Persian mathematician, al-Khowarizmi (ca. 825). (The word *algebra* is derived from the title of this book.)

As an adult, Fibonacci made frequent business trips to Egypt, Syria, Greece, France, and Constantinople, where he studied the various systems of arithmetic then in use, and exchanged views with native scholars. He also lived for a time at the court of the Roman Emperor, Frederick II (1194–1250), and engaged in scientific debates with the Emperor and his philosophers.

Around 1200, at the age of about 30, Fibonacci returned home to Pisa. He was convinced of the elegance and practical superiority of the Indo-Arabic system over the Roman numeration system then in use in Italy. In 1202, Fibonacci published his pioneering work, *Liber Abaci* (*The Book of the Abacus*.) (The word *abaci* here does not refer to the hand calculator called an abacus, but to computation in general.) *Liber Abaci* was devoted to arithmetic and elementary algebra; it introduced the Indo-Arabic numeration system and arithmetic algorithms to Europe. In fact, Fibonacci

Figure 1.1. Fibonacci (*Source*: David Eugene Smith Collection, Rare Book and Manuscript Library, Columbia University.).

demonstrated in this book the power of the Indo-Arabic system more vigorously than in any mathematical work up to that time. *Liber Abaci*'s 15 chapters explain the major contributions to algebra by al-Khowarizmi and another Persian mathematician, Abu Kamil (ca. 900). Six years later, Fibonacci revised *Liber Abaci* and dedicated the second edition to Michael Scott, the most famous philosopher and astrologer at the court of Frederick II.

After *Liber Abaci*, Fibonacci wrote three other influential books. *Practica Geometriae* (*Practice of Geometry*), written in 1220, is divided into eight chapters and is dedicated to Master Domonique, about whom little is known. This book skillfully presents geometry and trigonometry with Euclidean rigor and some originality. Fibonacci employs algebra to solve geometric problems and geometry to solve algebraic problems, a radical approach for the Europe of his day.

His next two books, the *Flos* (*Blossom* or *Flower*) and the *Liber Quadratorum* (*The Book of Square Numbers*) were published in 1225. Although both deal with number theory, *Liber Quadratorum* earned Fibonacci his reputation as a major number theorist, ranked between the Greek mathematician Diophantus (ca. 250 A.D.) and the French mathematician Pierre de Fermat (1601–1665). *Flos* and *Liber Quadratorum* exemplify Fibonacci's brilliance and originality of thought, which outshine the abilities of most scholars of his time.

In 1225 Frederick II wanted to test Fibonacci's talents, so he invited him to his court for a mathematical tournament. The contest consisted of three problems. The first was to find a rational number x such that both $x^2 - 5$ and $x^2 + 5$ are squares of rational numbers. Fibonacci gave the correct answer 41/12: $(41/12)^2 - 5 = (31/12)^2$ and $(41/12)^2 + 5 = (49/12)^2$.

The second problem was to find a solution of the cubic equation $x^3 + 2x^2 + 10x - 20 = 0$. Fibonacci showed geometrically that it has no solutions of the form $\sqrt{a + \sqrt{b}}$, but gave an approximate solution, 1.3688081075, which is correct to nine decimal places. This answer appears in the *Flos* without any explanation.

The third problem, also recorded in the *Flos*, was to solve the following:

> Three people share 1/2, 1/3, and 1/6 of a pile of money. Each takes some money from the pile until nothing is left. The first person then returns one- half of what he took, the second one-third, and the third one-sixth. When the total thus returned is divided among them equally, each possesses his correct share. How much money was in the original pile? How much did each person take from the pile?

Fibonacci established that the problem was indeterminate and gave 47 as the smallest answer. In the contest, none of Fibonacci's competitors could solve any of these problems.

The Emperor recognized Fibonacci's contributions to the city of Pisa, both as a teacher and as a citizen. Today, a statue of Fibonacci stands in a garden across the Arno River, near the Leaning Tower of Pisa.

Not long after Fibonacci's death in about 1240, Italian merchants began to appreciate the power of the Indo-Arabic system and gradually adopted it for business transactions. By the end of the sixteenth century, most of Europe had accepted it. *Liber Abaci* remained the European standard for more than two centuries and played a significant role in displacing the unwieldy Roman numeration system.

2

THE RABBIT PROBLEM

Fibonacci's classic book, *Liber Abaci*, contains many elementary problems, including the following famous problem on rabbits:

> Suppose there are two newborn rabbits, one male and the other female. Find the number of rabbits produced in a year if:
>
> 1) each pair takes one month to become mature;
>
> 2) each pair produces a mixed pair every month, from the second month on; and
>
> 3) no rabbits die during the course of the year.

Suppose, for convenience, that the original pair of rabbits was born on January 1. They take a month to become mature, so there is still only one pair on February 1. On March 1, they are two months old and produce a new mixed pair, a total of two pairs. Continuing like this, there will be three pairs on April 1, five pairs on May 1, and so on. See the last row of Table 2.1.

TABLE 2.1.

Number of Pairs	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Adults	0	1	1	2	3	5	8	13
Babies	1	0	1	1	2	3	5	8
Total	1	1	2	3	5	8	13	21

Figure 2.1. Lucas (*Source*: H. C. Williams, *Edouard Lucas and Primality Testing*, New York: Wiley, 1998. Copyright © 1998, reprinted with permission of John Wiley & Sons, Inc.).

FIBONACCI NUMBERS

The numbers in the bottom row are called *Fibonacci numbers*, and the number sequence 1, 1, 2, 3, 5, 8, ... is the *Fibonacci sequence*. Table A.2 in the Appendix lists the first 100 Fibonacci numbers.

The sequence was given its name in May of 1876 by the outstanding French mathematician François-Edouard-Anatole-Lucas (Fig. 2.1),* who had originally called it "the series of Lamé," after the French mathematician Gabriel Lamé (1795–1870). It is a bit ironic that despite Fibonacci's numerous mathematical contributions, he is primarily remembered for this sequence that bears his name.

*François-Edouard-Anatole-Lucas was born in Amiens, France, in 1842. After completing his studies at the École Normale in Amiens, he worked as an assistant at the Paris Observatory. He served as an artillery officer in the Franco-Prussian war and then became professor of mathematics at the Lycee Saint-Louis and Lycee Charlemagne, both in Paris, and he was a gifted and entertaining teacher. Lucas died of a freak accident at a banquet; his cheek was gashed by a shard that flew from a plate that was accidently dropped; he died from infection within a few days, on October 3, 1891.

Lucas loved computing and developed plans for a computer, but it never materialized. Besides his contributions to number theory, he is known for his four-volume classic on recreational mathematics. Best known among the problems he developed is the *Tower of Brahma*.

The Fibonacci sequence is one of the most intriguing number sequences, and it continues to provide ample opportunities for professional and amateur mathematicians to make conjectures and to expand the mathematical horizon.

The sequence is so important that an organization of mathematicians, *The Fibonacci Association*, has been formed for the study of Fibonacci and related integer sequences. The association was founded in 1963 by Verner E. Hoggatt, Jr. (1921–1980) of San Jose State College (now San Jose State University), California, and Brother Alfred Brousseau (1907–1988) of St. Mary's College in California. The association publishes *The Fibonacci Quarterly*, devoted to articles related to integer sequences.

A close look at the Fibonacci sequence reveals that it has a fascinating property: every Fibonacci number, except the first two, is the sum of the two immediately preceding Fibonacci numbers. (At the given rate, there will be 144 pairs of rabbits on December 1. This can be verified by extending Table 2.1 through December.)

RECURSIVE DEFINITION

This observation yields the following recursive definition of the nth Fibonacci number, F_n:

$$\begin{aligned} &F_1 = F_2 = 1 & & \leftarrow \text{Initial conditions} \\ &F_n = F_{n-1} + F_{n-2} & n \geq 3 \quad & \leftarrow \text{Recurrence relation} \end{aligned} \tag{2.1}$$

We shall formally establish the validity of this recurrence relation shortly.

It is not known whether Fibonacci knew of this relation. If he did, no record exists to that effect. In fact, the first written confirmation of the recurrence relation appeared four centuries later, when the great German astronomer and mathematician Johannes Kepler (1571–1630) wrote that Fibonacci must have surely noticed this recursive relationship. In any case, it was first noticed by the Dutch mathematician Albert Girard (1595–1632).

However, according to P. Singh of Raj Narain College in Bihar, India, Fibonacci numbers and the recursive formulation were known in India several centuries before Fibonacci proposed the problem; they were given by Virahanka (between 600 and 800 A.D.), Gopala (prior to 1135 A.D.), and Hemacandra (about 1150 A.D.). In fact, Fibonacci numbers also occur as a special case of a formula established by Narayana Pandita (1356 A.D.).

The growth of the rabbit population can be displayed nicely in a tree diagram, as Figure 2.2 shows. Each new branch of the "dream-tree" becomes an adult branch in one month and each adult branch, including the trunk, produces a new branch every month.

Table 2.1 shows several interesting relationships among the numbers of adult pairs, baby pairs, and total pairs. To see these relationships, let A_n denote the number of adult pairs and B_n the number of baby pairs in month n, where $n \geq 1$. Clearly, $A_1 = 0$, and $A_2 = 1 = B_1$.

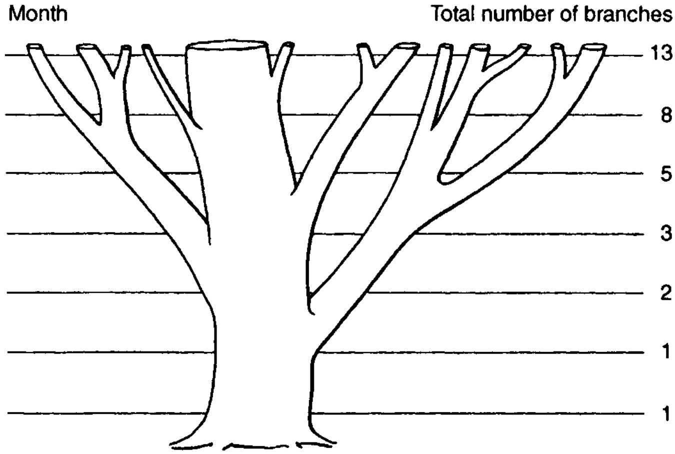


Figure 2.2. A Fibonacci tree.

Suppose $n \geq 3$. Since each adult pair produces a mixed baby pair in month n, the number of baby pairs in month n equals the number of adult pairs in the preceding month, that is, $B_n = A_{n-1}$. Then:

$$\begin{pmatrix}\text{Number of pairs} \\ \text{in month } n\end{pmatrix} = \begin{pmatrix}\text{Number of adult pairs} \\ \text{in month } n-1\end{pmatrix} + \begin{pmatrix}\text{Number of baby pairs} \\ \text{in month } n-1\end{pmatrix}$$

That is,

$$\begin{aligned} A_n &= A_{n-1} + B_{n-1} \\ &= A_{n-1} + A_{n-2} \qquad n \geq 3 \end{aligned}$$

Thus A_n satisfies the same recurrence relation as the Fibonacci recurrence relation (FRR), where $A_2 = 1 = A_3$. Consequently, $F_n = A_{n+1}, n \geq 1$.

Notice that:

$$\begin{pmatrix}\text{Total number of pairs} \\ \text{in month } n\end{pmatrix} = \begin{pmatrix}\text{Number of adult pairs} \\ \text{in month } n\end{pmatrix} + \begin{pmatrix}\text{Number of baby pairs} \\ \text{in month } n\end{pmatrix}$$

That is, $F_n = A_n + B_n = A_n + A_{n-1}$, where $n \geq 3$. Thus $F_n = F_{n-1} + F_{n-2}, n \geq 3$. This establishes the Fibonacci recurrence relation observed earlier.

Since $F_n = A_{n+1}$, where $n \geq 1$, every entry in row 1, beginning with the second element (February), is a Fibonacci number. In other words, the ith element in row 1 is F_{i-1}, where $i \geq 2$. Likewise, since $B_n = A_{n-1} = F_{n-2}$, where $n \geq 3$, the ith element in row 2 is F_{i-2}, where $i \geq 3$.

The recursive definition of F_n yields a straightforward method for computing it, as Algorithm 2.1 shows.

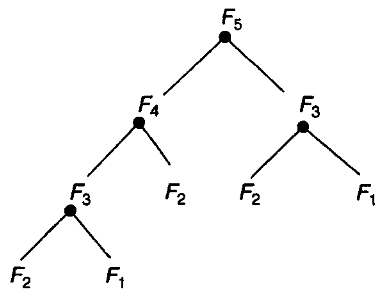


Figure 2.3. Tree diagram of recursive computing of F_5.

```
Algorithm Fibonacci(n)
(* This algorithm computes the nth Fibonacci number
using recursion. *)
Begin (* algorithm *)
  if n = 1 or n = 2 then (* base cases *)
    Fibonacci ← 1
  else                (* general case *)
    Fibonacci ← Fibonacci(n − 1) + Fibonacci(n − 2)
End (* algorithm *)
```

Algorithm 2.1.

The tree diagram in Figure 2.3 illustrates the recursive computing of F_5, where each dot represents an addition.

Using the recurrence relation (Eq. 2.1), we can assign a meaningful value to F_0. When $n = 2$, Eq. (2.1) yields $F_2 = F_1 + F_0$, that is, $1 = 1 + F_0$, so $F_0 = 0$. This fact will come in handy in our later discussions.

In the case of a nontrivial triangle, it is well known that the sum of the lengths of any two sides is greater than the length of the third side. Accordingly, the FRR can be interpreted to mean that *no three consecutive Fibonacci numbers can be the lengths of the sides of a nontrivial triangle.*

LUCAS NUMBERS

Using the Fibonacci recurrence relation and different initial conditions, we can construct new number sequences. For instance, let L_n be the nth term of a sequence with $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$, $n \geq 3$. The resulting sequence 1, 3, 4, 7, 11, ... is called the *Lucas sequence*, after Edouard Lucas; L_n is the nth term of the sequence. Table A.2 also lists the first 100 Lucas numbers.

We will see in later chapters that L_n and F_n are very closely related, and hence the title of this book. For instance, both L_n and F_n satisfy the same recurrence relation.

FIBONACCI AND LUCAS SQUARES AND CUBES

Of the infinitely many Fibonacci numbers, some have special characteristics. For example, only two distinct Fibonacci numbers are perfect squares, namely, 1 and 144. This was established in 1964 by J. H. E. Cohn of the University of London. In the same year, Cohn also established that 1 and 4 are the only Lucas squares (see Chapter 34).

In 1969, H. London of McGill University and R. Finkelstein of Bowling Green State University proved that there are exactly two distinct Fibonacci cubes, namely, 1 and 8, and that the only Lucas cube is 1.

A UBIQUITOUS FIBONACCI NUMBER AND ITS CONSTANT LUCAS COMPANION

Another Fibonacci number that appears to be ubiquitous is 89.

- Since 1/89 is a rational number, its decimal expansion is periodic:

$$\frac{1}{89} = 0.\overline{011235955056179775280\,(\underset{\uparrow}{89})\,887640449438202247191}$$

  The period is 44, and a surprising number occurs in the middle of a repeating block.
- It is the eleventh Fibonacci number, and both 11 (the fifth Lucas number) and 89 are prime numbers. While 89 can be viewed as the $(8+3)$rd Fibonacci number, it can also be looked at as the $(8 \cdot 3)$rd prime.
- Concatenating 11 and 89 gives the number 1189. Since $1189^2 = (1 + 2 + 3 + \cdots + 1681)/2$, it is also a triangular number. Interestingly enough, there are 1189 chapters in the Bible, of which 89 are in the four gospels.
- Eighty-nine is the smallest number to stubbornly resist being transformed into a palindrome by the familiar "reverse the digits and then add" method. In this case, it takes 24 steps to produce a palindrome, namely, 8813200023188.
- $8+9$ is the sum of the four primes preceding 11, and $8 \cdot 9$ is the sum of the four primes succeeding it: $17 = 2+3+5+7$ and $72 = 13+17+19+23$.
- The most recent year divisible by 89 is 1958: $1958 = 2 \cdot 11 \cdot 89$. Notice the prominent appearance of 11 again.
- The next year divisible by 89 is $2047 = 2^{11} - 1$. Again, 11 makes a conspicuous appearance. It is, in fact, the smallest number of the form $2^p - 1$, which is not a prime, where p is of course a prime. Primes of the form $2^p - 1$ are called *Mersenne primes*, after the French Franciscan priest Marin Mersenne (1588–1648), so 2047 is the smallest Mersenne number that is *not* a prime.

- On the other hand, $2^{89} - 1$ is a Mersenne prime; in fact, it is the tenth Mersenne prime, discovered in 1911 by R. E. Powers. Its decimal value contains 27 digits and looks like this:

$$2^{89} - 1 = 6189700196 \cdots 11$$

  The first three digits are significant because that they are the first three decimal digits of an intriguing irrational number we shall encounter in Chapters 20–27. Once again, note the surprising appearance of 11 at the end.
- Multiply the two digits of 89; add its digits again; and their sum is again 89: $(8 \cdot 9) + (8 + 9) = 89$. (It would be interesting to check if there are other numbers that exhibit this remarkable behavior.) Also, $8/9 \approx 0.89$.
- There are only two consecutive positive integers, one of which is a square and the other a cube: $8 = 2^3$ and $9 = 3^2$.
- Square the digits of 89 and add them to obtain 145. Add the squares of its digits again. Continue like this. After eight iterations, we return to 89:

$$89 \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58 \to 89$$

  In fact, if we apply this "sum the squares of the digits" method to any number, we will eventually attain 89 or 1.
- On 8/9 in 1974, an unfortunate and unprecedented event occurred in the history of the United States—the resignation of President Richard M. Nixon. Strangely enough, if we swap the digits of 89, we get the date on which Nixon was pardoned by his successor, President Gerald R. Ford.

All these fascinating observations about 11 and 89 were made in 1996 by M. J. Zerger of Adams State College, Colorado.

Soon after these Fibonacci curiosities appeared in *Mathematics Teacher*, G. J. Greenbury of England (private communication, 2000) contacted Zerger with two curiosities involving the decimal expansions of two primes:

$$\frac{1}{29} = 0.\overline{0344827586206(89)6551724137931}$$

$$\frac{1}{59} = 0.\overline{0169491525423728813559322022033(89)8305084745762711864406779661}$$

Curiously enough, 89 makes its remarkable appearance in the repeating block of each expansion.

R. K. Guy of the University of Calgary, Canada, in his fascinating book, *Unsolved Problems in Number Theory*, presents an interesting number sequence $\{x_n\}$. It has a quite remarkable and not immediately obvious relationship with 89. The sequence is defined recursively as follows:

$$x_0 = 1$$
$$x_n = \frac{1 + x_0^3 + x_1^3 + \cdots + x_{n-1}^3}{n}$$

For example, $x_0 = 1, x_1 = (1 + 1^3)/1 = 2$, and $x_2 = (1 + 1^3 + 2^3)/2 = 5$.
Surprisingly enough, x_n is integral for $0 \leq n < 89$, but x_{89} is not.

FIBONACCI AND PRIMES

Zerger also observed that the product $F_6 F_7 F_8 F_9$ is the product of the first seven prime numbers: $F_6 F_7 F_8 F_9 = 13 \cdot 21 \cdot 34 \cdot 55 = 510, 510 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$. Interestingly enough, 510 is the Dewey Decimal Classification Number for Mathematics.

FIBONACCI AND LUCAS PRIMES

Many Fibonacci and Lucas numbers are indeed primes. For example, the Fibonacci numbers 2, 3, 5, 13, 89, 233, and 1597 are primes, and so are the Lucas numbers 3, 7, 11, 29, 47, 199, and 521. Although it is widely believed that there are infinitely many Fibonacci and Lucas primes, their proofs still remain elusive.

The largest known Fibonacci prime is F_{9311}, and the largest known Lucas prime is L_{14449}. Discovered in 1999 by H. Dubner and W. Keller, they are 1946 and 3020 digits long, respectively. (Chapter 5 discusses a method for determining the number of digits in both F_n and L_n.)

Table A.3 lists the canonical prime factorizations of the first 100 Fibonacci numbers. Lucas had found the prime factorizations of the first 60 Fibonacci numbers before March 1877 and most likely even earlier. Boldface type in the table indicates the corresponding prime factor's first appearance in the list. For instance, the largest prime among the first 100 Fibonacci numbers is F_{83}.

Table A.4 gives the complete prime factorizations of the first 100 Lucas numbers.

CUNNINGHAM CHAINS

A *Cunningham chain*, named after Lt. Col. Allan J. C. Cunningham (1842–1928), an officer in the British Army, is a sequence of primes in which each element is one more than twice its predecessor. Interestingly enough, the smallest six-element chain begins with 89: 89, 179, 359, 719, 1439, 2879.

Are there Fibonacci and Lucas numbers that are one more than or one less than a square? A cube? We shall find the answers shortly.

FIBONACCI AND LUCAS NUMBERS $w^2 \pm 1, w \geq 0$

In 1973, R. P. Finkelstein of Bowling Green State University, Ohio, established yet another curiosity: The only Fibonacci numbers of the form $w^2 + 1$, where $w \geq 0$, are 1, 2, and 5: $1 = 0^2 + 1$, $2 = 1^2 + 1$, and $5 = 2^2 + 1$.

Two years later, Finkelstein proved that the only Lucas numbers of the same form are 2 and 1: $2 = 1^2 + 1$ and $1 = 0^2 + 1$.

In 1981, N. R. Robbins of Bernard M. Baruch College, New York, proved that the only Fibonacci numbers of the form $w^2 - 1$, where $w \geq 0$, are 3 and 8: $3 = 2^2 - 1$ and $8 = 3^2 - 1$. The only such Lucas number is 3.

FIBONACCI AND LUCAS NUMBERS $w^3 \pm 1, w \geq 0$

In the same year, Robbins also determined all Fibonacci and Lucas numbers of the form $w^3 \pm 1$, where $w \geq 0$. There are two Fibonacci numbers of the form $w^3 + 1$, namely, 1 and 2: $1 = 0^3 + 1$ and $2 = 1^3 + 1$. There are two Lucas numbers of the same form: 1 and 2.

There are no Fibonacci numbers of the form $w^3 - 1$, where $w \geq 0$. But there is exactly one such Lucas number, namely, 7: $7 = 2^3 - 1$.

FIBONACCI NUMBERS $(a^3 \pm b^3)/2$

Certain Fibonacci numbers can be expressed as one-half of the sum or difference of two cubes. For example, $1 = (1^3 + 1^3)/2$, $8 = (2^3 + 2^3)/2$, and $13 = (3^3 - 1^3)/2$. In fact, at the 1969 Summer Institute on Number Theory, held at Stony Brook, New York, H. M. Stark of the University of Michigan at Ann Arbor asked: Which Fibonacci numbers have this distinct property? This problem is linked to the finding of all complex quadratic fields with class 2. In 1983, J. A. Antoniadis tied such fields to solutions of certain diophantine equations.

FIBONACCI AND LUCAS TRIANGULAR NUMBERS

A *triangular number* is a positive integer of the form $n(n + 1)/2$. The first five triangular numbers are 1, 3, 6, 10, and 15; they can be represented geometrically, as Figure 2.4 shows.

In 1963, M. H. Tallman of Brooklyn, New York, observed that the Fibonacci numbers 1, 3, 21, and 55 are triangular numbers:

$$1 = \frac{1 \cdot 2}{2}, \qquad 3 = \frac{2 \cdot 3}{2}, \qquad 21 = \frac{6 \cdot 7}{2}, \qquad \text{and} \qquad 55 = \frac{10 \cdot 11}{2}$$

He asked if there were any other Fibonacci number that is also triangular.

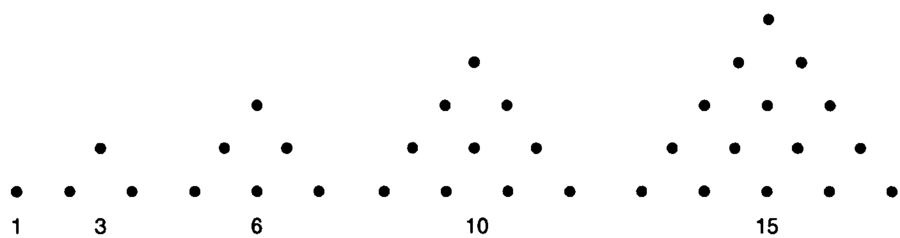


Figure 2.4. The first five triangular numbers.

Twenty-two years later, S. R. Wall of Trident Technical College, South Carolina, established that there are *no* other triangular numbers in the first one billion Fibonacci numbers. In fact, he conjectured that there are no other such Fibonacci numbers.

In 1976, Finkelstein proved that 1, 3, 21, and 55 are the only triangular Fibonacci numbers of the form F_{2n}.

In fact, eleven years later, L. Ming of Chongqing Teachers' College, China, proved conclusively that 1, 3, 21, and 55 are the only Fibonacci triangular numbers. This result is a byproduct of the two following results by Ming:

- $8F_n + 1$ is a perfect square if and only if $n = 0, \pm 1, 2, 4, 8, 10$.
- F_n is triangular if and only if $n = \pm 1, 2, 4, 8, 10$.

Are there Lucas numbers that are also triangular? Obviously, 1 and 3 are. In fact, in 1990, Ming also established that the only such Lucas numbers are 1, 3, and 5778:

$$1 = \frac{1 \cdot 2}{2}, \qquad 3 = \frac{2 \cdot 3}{2}, \qquad \text{and} \qquad 5778 = \frac{107 \cdot 108}{2}$$

FIBONACCI AND THE BEASTLY NUMBER

In 1989, C. Singh of St. Laurent's University in Quebec, Canada, discovered some mystical relationships between the infamous beastly number, 666, and Fibonacci numbers F_n:

- $666 = F_{15} + F_{11} - F_9 + F_1$, where $15 + 11 - 9 + 1 = 6 + 6 + 6$.
- $666 = F_1^3 + F_2^3 + F_4^3 + F_5^3 + F_6^3$, where the sum of the subscripts equals
$$1 + 2 + 4 + 5 + 6 = 6 + 6 + 6$$
- $666 = [F_1^3 + (F_2 + F_3 + F_4 + F_5)^3]/2$.

EXERCISES 2

1. Compute the first 20 Fibonacci numbers.
2. Compute the first 20 Lucas numbers.
3. Determine the value of L_0.
4. Using the FRR (Eq. 2.1), compute the value of F_{-n}, where $1 \leq n \leq 10$.
5. Using Exercise 4, predict the value of F_{-n} in terms of F_n.
6. Compute the value of L_{-n}, where $1 \leq n \leq 10$.
7. Using Exercise 6, predict the value of L_{-n} in terms of L_n.

To commemorate the publication of the maiden issue of the *Journal of Recreational Mathematics*, L. Bankoff of Los Angles published his discovery that $F_{20} - F_{19} - F_{15} - F_5 - F_1 = F_{17} + F_{13} + F_{11} + F_9 + F_7 + F_3$ and that each sum gives the year.

8. Find the year in which the journal was first published.
9. Verify that the sums of the subscripts of the Fibonacci numbers on either side are equal.

Compute the sum $\sum_1^n F_i$ for each value of n.

10. 3
11. 5
12. 7
13. 8
14. Using Exercises 10–13, predict a formula for $\sum_1^n F_i$.

15–18. Compute the sum $\sum_1^n L_i$ for each value of n in Exercises 10–13.

19. Using Exercises 15–18, predict a formula for $\sum_1^n L_i$.

20–23. Compute the sum $\sum_1^n F_i^2$ for each value of n in Exercises 10–13.

24. Using Exercises 20–23, predict a formula for the sum $\sum_1^n F_i^2$.

25–28. Compute the sum $\sum_1^n L_i^2$ for each value of n in Exercises 10–13.

29. Using Exercises 20–23, predict a formula for the sum $\sum_1^n L_i^2$.
30. Verify that $F_{2n} = F_n L_n$ for $n = 3$ and $n = 8$.
31. Verify that $L_n = F_{n-1} + F_{n+1}$ for $n = 4$ and $n = 7$.

Let a_n denote the number of additions needed to compute F_n by recursion:

32. Define a_n recursively.

33. Show that $a_n = F_n - 1, n \geq 1$.
34. Prove that $F_n < 1.75^n$ for every positive integer n (LeVeque, 1962).
35. Show that there are no four distinct Fibonacci numbers in arithmetic progression (Silverman, 1964).
36. Let $I_n = \int_0^1 x^{I_{n-1}} dx$, where $n \geq 2$ and $I_1 = \int_0^1 x\, dx$. Evaluate I_n (Lind, 1965).
37. If $F_n < x < F_{n+1} < y < F_{n+2}$, then $x + y$ cannot be a Fibonacci number (Hoggatt, 1982).

Suppose we introduce a mixed pair of 1-month-old rabbits into a large enclosure on the first day of a certain month. By the end of each month, the rabbits become mature and each pair produces $k-1$ mixed pairs of offspring at the beginning of the following month. (*Note:* $k \geq 2$.) For instance, at the beginning of the second month, there is one pair of 2-month-old rabbits and $k - 1$ pairs of 0-month-olds; at the beginning of the third month, there is one pair of 3-month-olds, $k - 1$ pairs of 1-month-olds, and $k(k - 1)$ pairs of 0-month-olds. Assume the rabbits are immortal. Let a_n denote the average age of the rabbit-pairs at the beginning of the nth month (Filipponi and Singmaster, 1990).

**38. Define a_n recursively.
**39. Predict an explicit formula for a_n.
**40. Prove the formula in Exercise 39.
41. (For those familiar with the concept of limits) Find $\lim_{n\to\infty} a_n$.

3

FIBONACCI NUMBERS IN NATURE

> Come forth into the light of things,
> let Nature be your teacher.
>
> —William Wordsworth

Interestingly enough, the amazing Fibonacci numbers occur in quite unexpected places in nature.

FIBONACCI AND THE EARTH

Do Fibonacci numbers also appear elsewhere? Zerger observed that the equatorial diameter of the earth in miles is approximately the product of two alternate Fibonacci numbers, and that this in kilometers is approximately the product of three consecutive Fibonacci numbers:

$$55 \cdot 144 = 7920\,\text{miles} \quad \text{and} \quad 89 \cdot 144 = 12{,}816\,\text{kilometers}$$

For the curious-minded, the earth's diameter, according to *The 2000 World Almanac and Book of Facts*, is 7928 miles and 12,756 kilometers; the polar diameter is 7901 miles. The diameter of Jupiter, the largest planet, is 11 times that of the earth.

FIBONACCI AND ILLINOIS

In 1992, Zerger discovered some astonishing occurrences of Fibonacci numbers in relation to the state of Illinois:

- Illinois was admitted to the Union on the 3rd of December.
- Illinois is the fifth largest state, according to the 1990 census.
- Illinois' name consists of 8 letters.
- Illinois is the thirteenth state, when the states are arranged alphabetically.
- Illinois was the twenty-first state admitted to the Union. The postal abbreviation IL is formed with the ninth and twelfth letters: $9 + 12 = 21$.
- Interstate 55 begins in Chicago and roughly follows the 89th parallel to New Orleans.

FIBONACCI AND FLOWERS

The number of petals in many flowers is often a Fibonacci number. For instance, count the number of petals in the flowers pictured in Figure 3.1. Enchanter's nightshade has two petals, iris and trillium three, wild rose five, and delphinium and cosmos eight. Most daisies have 13, 21, or 34 petals; there are even daisies with 55 and 89 petals. Table 3.1 lists the Fibonacci number of petals in an assortment of flowers. Although some plants, such as buttercup and iris, always display the same number of petals, some do not. For example, delphinium blossoms sometimes have 5 petals and sometimes 8 petals, and some Michaelmas daisies have 55 petals, while some have 89 petals.

The cross section of an apple reveals a pentagonal shape with five pods. The starfish, with five limbs, also exhibits a Fibonacci number (see Fig. 3.2).

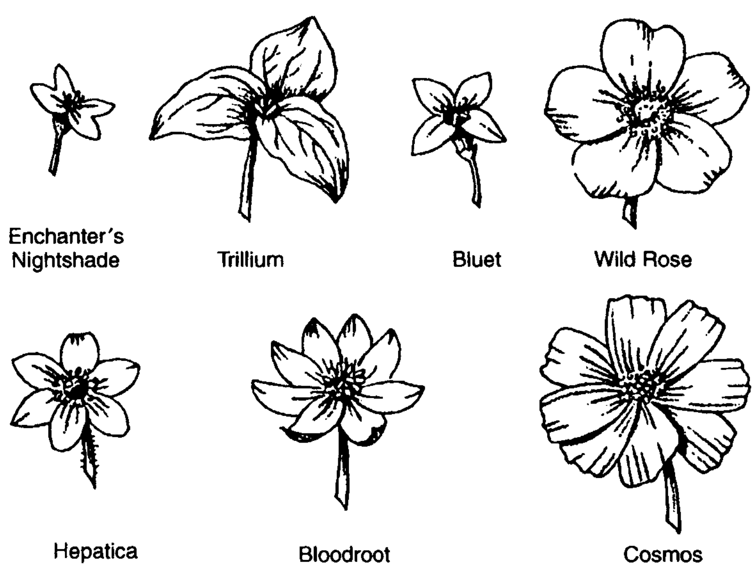
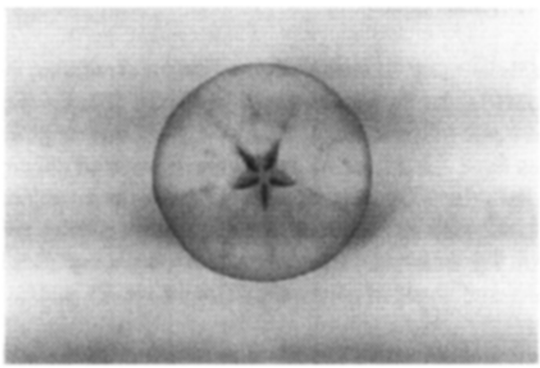


Figure 3.1. Flowers.

TABLE 3.1.

Plant	Number of Petals
Enchanter's nightshade	2
Iris, lilly	3
Buttercup, columbine, delphinium, larkspur, wall lettuce	5
Celandine, delphinium, field senecio, squalid senecio	8
Chamomile, cineraria, corn marigold, double delphinium, globeflower	13
Aster, black-eyed Susan, chicory, doronicum, helenium, hawkbit	21
Daisy, gailliardia, plantain, pyrethrum, hawkweed	34



(a)

(b)

Figure 3.2. (*a*) Cross section of an apple; (*b*) Starfish.

FIBONACCI AND TREES

Fibonacci numbers are also found in some spiral arrangements of leaves on the twigs of plants and trees. From any leaf on a branch, count up the number of leaves until you reach the leaf directly above it; the number of leaves is often a Fibonacci number. On basswood and elm trees, this number is 2; on beech and hazel trees, it is 3; on apricot, cherry, and oak trees, it is 5; on pear and poplar trees, it is 8; and on almond and willow trees, it is 13 (see Fig. 3.3).

Here is another intriguing fact: The number of turns, clockwise or counterclockwise, we can take from the starting leaf to the terminal leaf is also usually a Fibonacci number. For example, on basswood and elm trees, it takes one turn; for beech and hazel trees, it is also 1; for apricot, cherry, and oak trees, it is 2; for pear and poplar trees, it is 3; and on almond and willow trees, it is 5.

The arrangement of leaves on the branches of *phyllotaxis.** Accordingly, the ratio of the number of turns to the number of leaves is called the *phyllotactic ratio* of the tree. Thus, the phyllotactic ratio of basswood and elm is 1/2; for beech and hazel, it is 1/3; for apricot, cherry, and oak, it is 2/5; for pear and poplar, it is 3/8; and for almond and willow, it is 5/13. These data are summarized in Table 3.2. As an example, it takes 3/8 of a full turn to reach from one leaf to the next leaf on a pear tree.

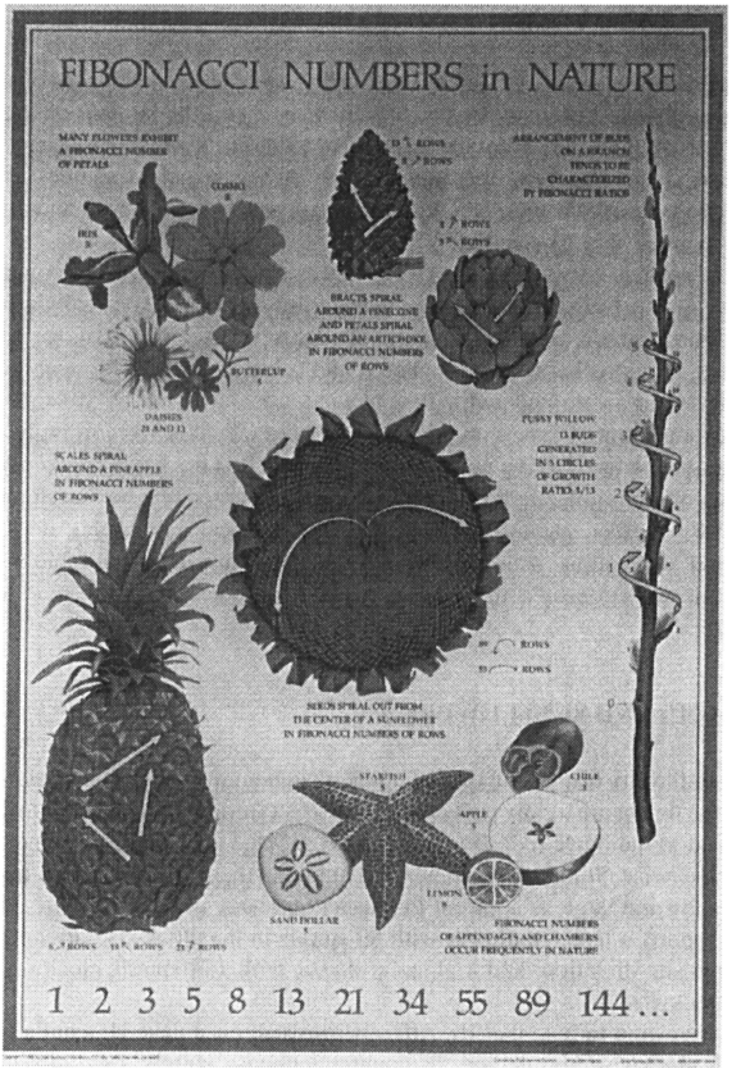
FIBONACCI AND SUNFLOWERS

Mature sunflowers display Fibonacci numbers in a unique and remarkable way. The seeds of the flower are tightly packed in two distinct spirals, emanating from the center of the head to the outer edge (Figs. 3.4 and 3.5). One goes clockwise and the other counterclockwise. Studies have shown that although there are exceptions, the number of spirals, by and large, is adjacent Fibonacci numbers; usually, they are 34 and 55. Hoggatt reports a large sunflower with 89 spirals in the clockwise direction and 55 in the opposite direction, and a gigantic flower with 144 spirals clockwise and 89 counterclockwise.

It is interesting to note that Br. Alfred Brousseau once gave Hoggatt a sunflower with 123 clockwise spirals and 76 counterclockwise spirals, two adjacent Lucas spirals.

In 1951, John C. Pierce of Goddard College in Massachusetts reported in *The Scientific Monthly* that the Russians had grown a sunflower head with 89 and 144 spirals. After reading his article on Fibonacci numbers, Margaret K. O'Connell and Daniel T. O'Connell of South Londonderry, Vermont, examined their sunflowers, raised from seeds from Burpee's. They found heads with 55 and 89 spirals, some with 89 and 144 spirals, and one giant head with 144 and 233 spirals. The latter seems to be a world record.

*The word *phyllotaxis* is derived from the Greek words *phyllon*, meaning *leaf*, and *taxis*, meaning *arrangement*.



From *Fibonacci Numbers in Nature (poster).* Copyright © 1988 by Dale Seymour Publications. Used with permission of Pearson Education.

FIBONACCI, PINECONES, ARTICHOKES, AND PINEAPPLES

The scale patterns on pinecones, artichokes, and pineapples provide excellent examples of Fibonacci numbers. The scales are in fact modified leaves closely packed on short stems, and they form two sets of spirals, called *parastichies.** Some spirals

*The word *parastichies* is derived from the Greek words *para*, meaning *beside* and *stichos*, meaning *row*.