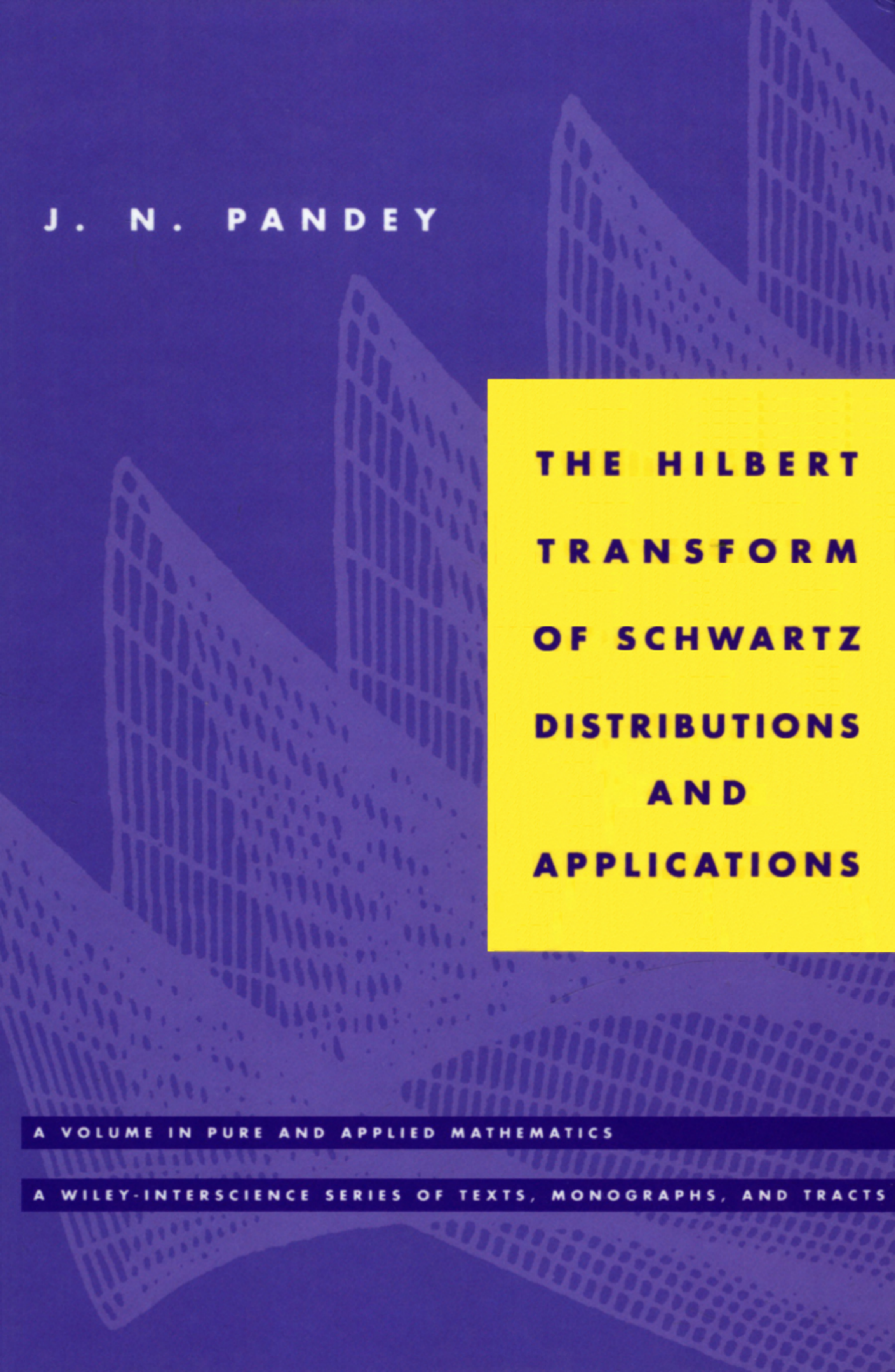

J. N. PANDEY

THE HILBERT TRANSFORM OF SCHWARTZ DISTRIBUTIONS AND APPLICATIONS

A VOLUME IN PURE AND APPLIED MATHEMATICS

A WILEY-INTERSCIENCE SERIES OF TEXTS, MONOGRAPHS, AND TRACTS

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J. N. PANDEY
Carleton University

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To my parents
(Pandit Chandrika Pandey and Shrimati Chameli Devi),
as well as to all of the 329 passengers and crew members
of Air India flight number 182 which crashed on June 23, 1985
near the Irish coast.

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PREFACE

For the past two decades I have been researching the Hilbert transform of Schwartz distributions. I and my colleagues have arrived at many new results. These results form the basis of this book which will be of interest not only to mathematicians but also to engineers and applied scientists. My objective is to demonstrate the wide applicability of Hilbert transform techniques. This book may be used either as a graduate-level textbook on the Hilbert transform of Schwartz distributions and periodic distributions or as a research monograph.

The Hilbert transform $(Hf)(x) = \frac{1}{\pi}(P)\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt$ arises in many fields such as

i. Signal processing (the Hilbert transform of periodic functions)
ii. Metallurgy (Griffith crack problem and the theory of elasticity)
iii. Dirichlet boundary value problems (potential theory)
iv. Dispersion relation in high energy physics, spectroscopy, and wave equations
v. Wing theory
vi. The Hilbert problem
vii. Harmonic analysis

The Hilbert problem during the last four decades has received considerable attention in metallurgical problems, namely in the Griffith crack problem in the theory of elasticity. Sneddon and Lowengrub who have been pioneers of applying the finite Hilbert transform in the theory of elasticity state: "The major development of the present century in the field of two-dimensional elasticity has been Muskhelishvili's work on the complex form of the two-dimensional equations due to G. B. Kolsov." Consequently a fair amount of treatment of classical as well as distributional Hilbert problems has been incorporated in the book. In particular, Chapter 2 is devoted to the classical Hilbert problem, whereas Chapter 3 and Chapter 6 cover distributional Hilbert problems.

The singular nature of the kernel $\frac{1}{\pi(x-t)}$ of the Hilbert transform has made the work on the Hilbert transform very difficult to accomplish and in turn the work on the Hilbert transform of distributions has suffered. Nevertheless, the problem of the Hilbert transform of distributions has received the attention of many mathematicians who had started working on the Hilbert transform of various subspaces of Schwartz distributions. Among them are Laurent Schwartz [87], Gel'fand and Shilov [44], Horvath [5], Bremermann [9], Jones [53], Lauwerier [58], Tillmann [97, 63], Beltrami and Wohlers [6], Orton [72], Mitrovic [61, 62, 63], and Carmichael [15, 16]. The approach to the Hilbert transform of distributions that I have developed with my colleagues is the simplest and the most effective. It is easily accessible to applied scientists despite the fact that I have used a fairly advanced treatment in this book.

Among many new results that I wish to point out are the inversion formula for the n-dimensional Hilbert transform $H^2 f = (-1)^n f$, $n > 1$ and a new definition for the Hilbert transform of periodic functions with period 2τ:

$$(Hf)(x) = \frac{1}{\pi} \lim_{N\to\infty} (P) \int_{-N}^{N} \frac{f(t)}{x-t} dt \tag{i}$$

$$= \frac{1}{2\tau} (P) \int_{-\tau}^{\tau} f(x-t) \cot\left(\frac{t\pi}{2\tau}\right) dt \tag{ii}$$

This identity is true at least for the class of functions $f \in L^p_{2\tau}$. My definition of the Hilbert transform of periodic functions is a generalization of the Hilbert transform of periodic functions with period 2π, defined as

$$(Hf)(x) = \frac{1}{2\pi} (P) \int_{-\pi}^{\pi} f(x-t) \cot\frac{t}{2}\, dt \tag{iii}$$

Definition (iii) was widely used by Butzer, Nessel, Oppenheim, Schaefer, and many others, and to the best of my knowledge, there has been no formula or definition for the Hilbert transform of periodic functions with period other than 2π. I also believe that the definition of the Hilbert transform of periodic functions in the form (i) will be especially useful to people working in signal processing for computational purposes. From definition (i), which is the definition for the Hilbert transform of functions, a unified theory of the Hilbert transform of periodic as well as nonperiodic functions can be developed.

In Chapter 7 I develop the theory of the Hilbert transform of periodic distributions and also the approximate Hilbert transform of periodic distributions. I use this theory to find a harmonic function $U(x, y)$ which is periodic in x with period 2τ vanishes as $y \to \infty$, uniformly $\forall x \in \mathbb{R}$ and tends to a periodic distribution f (with period 2τ) as $y \to 0^+$, in the weak distributional sense. The uniqueness of the solution is also proved.

My discussion proceeds from a Paley-Wiener type of theorem (Theorem 6.18) which gives the characterization of functions or generalized functions whose Fourier transform vanishes over certain orthants or the union of orthants of $\mathbb{R}^n$.

In Chapter 5 I also give a generalization of the Hilbert problem

$$F_+(x) - F_-(x) = f(x)$$

in higher dimensions and solve it. In Section 6.7 I calculate the p-norm $\|H\|_p$ of the Hilbert transform operator $H: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $p > 1$. In Theorem 6.3 I give a characterization of bounded linear operators on $L^p(\mathbb{R}^n)$, $p > 1$, which commute with translation and dilatation.

Another highlight of the book is the very elegant treatment of the one-dimensional Hilbert transform of distributions in D'_{L^p}, $p > 1$, in Chapter 3. Chapter 3 will be especially useful to applied scientists.

The book assumes that the reader has a background in the elements of functional analysis. Chapter 1 essentially deals with the prerequisite materials for the theory of distributions and Fourier transform.

Chapter 2 presents the Riemann-Hilbert problem and gives the background material to the study of the Hilbert transform. It includes sections on the appearance of the Hilbert transform in wing theory, in the theory of elasticity, in spectroscopy, and in high-energy physics.

Chapter 3 discusses the Hilbert transform of Schwartz distributions in D'_{L^p} and related boundary value problems.

Chapter 4 considers the Hilbert transform of Schwartz distributions in D'. It also discusses a Gel'fand and Shilov technique for the Hilbert transform of generalized functions and an improvement to their techniques.

Chapter 5 deals with n-dimensional Hilbert transform and the approximation technique in evaluating the Hilbert transform and the inversion formulas. The Hilbert transform of distribution in $D'_{L^p}(\mathbb{R}^n)$ is also covered, and many applications are given.

Chapter 6 considers the applications of the Hilbert transform to Riemann-Hilbert problems (classical as well as distributional). Many other related results are presented. One among many is the derivation of a Paley-Wiener theorem.

Chapter 7 deals with the periodic distributions and their Hilbert transforms.

With the firm belief that perfection never comes without practice I have included numerous examples in every chapter.

I wish to acknowledge the assistance of Professor E. L. Koh of the University of Regina, and of Professor S. A. Naimpally and Dr. James Bondar of my department who were very kind and patient in going through various chapters of the manuscript and gave me very useful suggestions. I want to express my sincere gratitude to Professor Angelo Mingarelli of my department who very patiently entered the graphic designs on my manuscript and helped me consult CDRAM (Math Reviews) for the preparation of the manuscript.

I further wish to thank Mr. Andrew E. Dabrowski a former student at Carleton, Mr. Sanjay Varma of the Mehta Research Institute Allahabad India, Ms. Nalini Sreeshylan of the Institute of Sciences Bangalore India, Mr. K. P. Sivaraman of the Tata Institute of Fundamental Research, Bombay, Mrs. Shelly Bereznin, Mr. Ibrahim Farah, and Mrs. Diane Berezowski of Carleton University for their help in the typing of the manuscript in its various forms.

The major part of the typing was done by Mrs. Diane Berezowski who modified all the chapters typed by others and unified them into a single TEX scheme along with her own typing. She never lost her temper despite the many changes I had asked her to incorporate in the manuscript. I am grateful to her for her patience and for her active and prompt cooperation.

In addition I would like to thank Drs. Q. M. Tariq and Ehab Bassily of my department who very patiently prepared the Index and the Notation Index of this book.

I also wish to acknowledge a grant from the Natural Sciences and Engineering Research Council of Canada in support of my research. I wish to thank the Tata Institute of Fundamental Research, Bombay, the Indian Institute of Science, Bangalore, and the Mehta Research Institute, Allahabad, for financially supporting my visits to their institutions, where a considerable part of the research work on this monograph was completed. In particular, I express my gratitude to Professor H. S. Mani, Director, Mehta Research Institute of Mathematics and Mathematical Physics for encouraging me to use the facilities of his institute.

My debt to my wife, Krishna for her constant support and cheerfulness under difficult circumstances when the manuscript was under preparation, is so profound as to defy description.

J. N. PANDEY

Ottawa, Canada

1

SOME BACKGROUND

1.1. FOURIER TRANSFORMS AND THE THEORY OF DISTRIBUTIONS

This chapter discusses some very important properties of the Fourier transform of functions that will be useful in developing the theory of the Fourier transform of distributions. It also develops some basic results concerning topological vector spaces, in particular, locally convex spaces, and extends these results to develop a theory of distributions and tempered distributions.

Definition. Let f be a function of a real variable t defined on the real line. Then its Fourier transform $F(w)$ is defined by the relation

$$(\mathcal{F}f)(w) = F(w) = \int_{-\infty}^{\infty} f(t)e^{iwt}dt \tag{1.1}$$

provided that the integral exists.

There are many variations on definition (1.1). Some authors add the factor $\frac{1}{2\pi}$ or $\frac{1}{\sqrt{2\pi}}$ outside the integral sign, and some take the kernel of the Fourier transform as e^{-iwt} in place of the kernel e^{iwt}. Some authors including L. Schwartz have written the kernel of the Fourier transform $e^{2\pi iwt}$. But these variations matter little.

The inverse Fourier transform of f in our case will be defined as

$$\mathcal{F}^{-1}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w)e^{-iwt}dw \tag{1.2}$$

provided that the integral exists.

Example 1. Let

$$f(t) = \begin{cases} 1, & -1 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Then

$$(\mathcal{F}f)(w) = \int_{-1}^{1} 1e^{itw}dt = \begin{cases} \dfrac{2\sin w}{w}, & w \neq 0 \\ 2 & \text{when } w = 0 \end{cases}$$

Note that the function $f(t) \in L^1$ but $(\mathcal{F}f)(w) \notin L^1$.

Theorem 1. Let $f \in L^1$. Then

i. $F(w) = \int_{-\infty}^{\infty} f(t)e^{iwt}dt$ is well defined $\forall\, w \in \mathbb{R}$.
ii. $F(w)$ is uniformly continuous and bounded on $\mathbb{R}$.
iii. $F(w) \to 0$ as $|w| \to \infty$.

Proof. (i) $\int_{-\infty}^{\infty} |f(t)e^{iwt}|\, dt \leq \int_{-\infty}^{\infty} |f(t)|\, dt$. Clearly $f(t)e^{iwt}$ is a measurable function of t. Therefore $f(t)e^{iwt}$ is absolutely integrable, and it is integrable for each $w \in \mathbb{R}$. Hence $F(w)$ is well defined for each $w \in \mathbb{R}$.

(ii) $|F(w)| \leq \int_{-\infty}^{\infty} |f(t)e^{iwt}|\, dt \leq \int_{-\infty}^{\infty} |f(t)|\, dt$. $F(w)$ is uniformly bounded. Now we prove that $F(w)$ is uniformly continuous on R. Choosing N large enough so that for an arbitrary $\epsilon > 0$, we have

$$\int_{N}^{\infty} |f(t)|\, dt + \int_{-\infty}^{-N} |f(t)|\, dt < \frac{\epsilon}{4} \tag{1.3}$$

A simple calculation shows that

$$\begin{aligned} \Delta F &= F(w + \Delta w) - F(w) \\ &= \int_{-N}^{N} f(t)[e^{i(w+\Delta w)t} - e^{iwt}]\, dt \\ &\quad + \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) f(t)e^{iwt}[e^{i\Delta wt} - 1]\, dt \end{aligned} \tag{1.4}$$

Now denote the first integral in the right hand side of (1.4) by I and the second pair of integrals by J:

$$|I| \leq \int_{-N}^{N} |f(t)|\, |e^{i\Delta wt} - 1|\, dt$$

By virtue of the uniform continuity of $(e^{i\Delta wt} - 1)$, we can choose δ small enough so that

$$|I| < \frac{\epsilon}{2} \quad \text{whenever} \quad |\Delta w| < \delta \tag{1.5}$$

δ being independent of t (and w as well).

$$|J| \le \left(\int_{-\infty}^{\infty} + \int_{-\infty}^{-N}\right) 2|f(t)|\,dt \le \frac{\epsilon}{2} \tag{1.6}$$

Combining (1.5) and (1.6), we have

$$|\Delta F| \le \epsilon \quad \text{whenever} \quad |\Delta w| < \delta$$

This proves the uniform continuity of $F(w)$ over the real line.

(iii) The space $\mathcal{D}(\mathbb{R}^n)$ of infinitely differentiable functions with compact support on $\mathbb{R}^n$ is dense in $L^p(\mathbb{R}^n)$ $p \ge 1$, and the identity map from $\mathcal{D}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ is continuous [67, 101]. Let now $\varphi \in \mathcal{D}(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} |f(t) - \varphi(t)|\,dt < \frac{\epsilon}{2}.$$

Then

$$F(w) = \int_{-\infty}^{\infty} [f(t) - \varphi(t)]e^{iwt}\,dt + \int_{-\infty}^{\infty} \varphi(t)e^{iwt}\,dt \tag{1.7}$$

Denoting the two integrals in the right hand side of (1.7) by J_1 and J_2, respectively, we see that

$$|J_1| \le \int_{-\infty}^{\infty} |f(t) - \varphi(t)|\,dt < \frac{\epsilon}{2} \tag{1.8}$$

A simple integration by parts shows that

$$J_2 \to 0 \text{ as } |w| \to \infty$$

Therefore there exists a $k > 0$ such that

$$|J_2| < \frac{\epsilon}{2}\ \forall\ |w| > k \tag{1.9}$$

Combining (1.8) and (1.9), we get for $\epsilon > 0$, that there exists a constant $k > 0$ such that $|F(w)| < \epsilon\ \forall\ |w| > k$. Since ϵ is arbitrary our result is proved. □

Theorem 2. Let $f \in L^1$ and $F(w)$ be the Fourier transform of f. Assume that $F(w) \in L^1$. Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{-iwt}dw \qquad \text{a.e.} \tag{1.10}$$

The equality (1.10) holds at all points of continuity of f. Proof of this inversion formula for the Fourier transform can be found in many books on integral transforms.

The Fourier transform of a function $f(t)$ defined from $\mathbb{R}^n \to \mathbb{R}$ is defined as $F(w) = \int_{\mathbb{R}^n} f(t)e^{it\cdot w}\,dt$, provided that the integral exists. Here $t = (t_1, t_2, \ldots, t_n)$ and

$w = (w_1, w_2, \ldots, w_n)$, $t \cdot w = t_1 w_1 + t_2 w_2 + \cdots + t_n w_n$. Theorems analogous to Theorems 1 and 2 are valid. The inversion formula analogous to (1.10) is

$$f(t) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} F(w) e^{-iw \cdot t} dw \qquad \text{a.e.}$$

which is also valid if f and F both $\in L^1(\mathbb{R}^n)$.

1.2. FOURIER TRANSFORMS OF L^2 FUNCTIONS

1.2.1. Fourier Transforms of Some Well-known Functions

Consider

$$\begin{aligned} \mathcal{F}(e^{-|t|}) &= \int_{-\infty}^{\infty} e^{-|t|} e^{iwt}\, dt \\ &= 2 \int_0^{\infty} \cos wte^{-t} \\ &= \frac{2}{w^2 + 1} \\ \mathcal{F}[h(t-1) - h(t-2)] &= \int_1^2 e^{iwt}\, dt \end{aligned}$$

where $h(t)$ is Heaviside's unit function,

$$\begin{aligned} &= \frac{e^{iwt}}{iw}\Bigg|_1^2 = \left[\frac{e^{2iw} - e^{iw}}{iw}\right] \\ \mathcal{F}\left[\frac{1}{1+t^2}\right] &= \int_{-\infty}^{\infty} \frac{e^{itw}}{1+t^2}\, dt \\ &= \pi[e^{-w} h(w) + e^{+w} h(-w)]. \end{aligned}$$

In a different category from the above is

$$\mathcal{F}(1) = \int_{-\infty}^{\infty} e^{iwt}\, dt$$

which does not exist in the classical sense but does exist in the distributional sense, as will be proved later.

We can verify that our inversion formula as stated before is valid in the case of these functions. For example,

$$\mathcal{F}^{-1}[e^{-w} h(w) + e^{w} h(-w)]\pi = \frac{1}{2\pi} \int_0^{\infty} \pi e^{-w} e^{-iwt} dw + \frac{1}{2\pi} \pi \int_{-\infty}^{0} e^{w} e^{-iwt} dw$$

$$= \frac{1}{2}\left[\frac{1}{1+it} + \frac{1}{1-it}\right]$$

$$= \frac{1}{1+t^2}$$

Theorem 3. Let $f(t)$ be continuous, and let $f'(t)$ be piecewise continuous on the real line such that $\lim_{|t|\to\infty} f(t) = 0$, and $f(t)$ is Fourier transformable $\forall\, w \in \mathbb{R}$. Then $f'(w)$ is also Fourier transformable $\forall\, w \in \mathbb{R}$, and

$$\mathcal{F}(f')(w) = (-iw)(\mathcal{F}f)(w)$$

Proof. Consider the operations

$$\begin{aligned}\mathcal{F}(f')(w) &= \int_{-\infty}^{\infty} f'(t)e^{iwt}\,dt \\ &= e^{iwt}f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} iwe^{iwt}f(t)\,dt \\ &= (-iw)\int_{-\infty}^{\infty} e^{iwt}f(t)\,dt \\ &= (-iw)\mathcal{F}(f)(w)\end{aligned}$$

These operations can be justified by integrating between $-M$ and N and letting $M, N \to \infty$. □

Corollary 1. If f' is continuous on $\mathbb{R}$ and is Fourier transformable, and if $\lim_{|t|\to\infty} f(t) = 0$, then $f(t)$ is also Fourier transformable and $(\mathcal{F}f')(w) = (-iw)(\mathcal{F}f)(w)$.

Corollary 2. If $f^{(n)}(t)$ is Fourier transformable and is continuous such that $\lim_{t\to\pm\infty} f^{(k)}(t) = 0$ for $k = 0, 1, 2, \ldots, n-1$, then $f^{(n-1)}, f^{(n-2)}, \ldots, f', f$ are all Fourier transformable and

$$\mathcal{F}(f^{(k)})(w) = (-iw)^k(\mathcal{F}f)(w), \qquad k = 1, 2, \ldots, n$$

Corollary 3. If f is continuously differentiable up to order n such that $\lim_{|t|\to\infty} f^{(k)}(t) = 0$ for each $k = 0, 1, 2, \ldots, n-1$ and $f(t)$ is Fourier transformable, then each of the derivatives $f', f'', \ldots, f^{(n)}$ is Fourier transformable and

$$\mathcal{F}(f^{(k)})(w) = (-iw)^k(\mathcal{F}f)(w), \qquad k = 1, 2, 3, \ldots, n.$$

Corollary 4. If $f, f', f'', \ldots, f^{(n-1)}$ are all continuous and $f^{(n)}$ is piecewise continuous in any aribtrary, finite closed interval of $\mathbb{R}$ and if $\lim_{|t|\to\infty} f^{(k)}(t) = 0$ for each $k = 0, 1, 2, \ldots, n-1$ and $f(t)$ is Fourier transformable, then $f^{(k)}(t)$ is Fourier transformable and $\mathcal{F}(f^{(k)})(w) = (-iw)^k(\mathcal{F}f)(w)$ for each $k = 1, 2, 3, \ldots, n$.

For functions defined over a finite measure space, every $f \in L^2(X)$ belongs to $L^1(X)$, but this result is not true here in general. Let us consider

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

This function $f \notin L^1(\mathbb{R})$, but it does belong to $L^2(\mathbb{R})$. There are functions that belong to $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ as well. For example $e^{-|t|} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

The Fourier transform of functions belonging to $L^2(\mathbb{R})$ does not necessarily exist in general in the pointwise sense. Also, if $f \in L^2(\mathbb{R})$, then the truncated function $f(t)\chi_{[-a,a]} \to f(t)$ in $L^2(\mathbb{R})$ as $a \to \infty$. Since $f(t)\chi_{[-a,a]} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the space of functions belonging to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ forms a dense subset of $L^2(\mathbb{R})$. The question now arises as to how the Fourier transform of $f \in L^2(\mathbb{R})$ is to be defined.

Using the above-mentioned density property, Plancherel proved the following well-known theorem [3, p. 91], which is called the *Plancheral theorem.*

Theorem 4. Let $f \in L^2(\mathbb{R})$. Then there exists a function $\hat{f}(w) \in L^2(\mathbb{R})$ such that

$$\left\| \hat{f}(w) - \int_{-a}^{a} f(t)e^{iwt}dt \right\|_2 \to 0 \qquad \text{as } a \to \infty \tag{1.11}$$

that is,

$$\hat{f}(w) = \underset{a\to\infty}{\text{l.i.m.}} \int_{-a}^{a} f(t)e^{iwt}dt$$

Moreover

$$\left\| f(x) - \frac{1}{2\pi}\int_{-a}^{a} \hat{f}(w)e^{-iwx}dw \right\|_2 \to 0 \qquad \text{as } a \to \infty \tag{1.12}$$

that is,

$$f(x) = \underset{a\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-a}^{a} \hat{f}(w)e^{-iwx}dw$$

For a measure theoretic and modern proof see Rudin [84] on the real and complex analysis.

It is further proved that

$$\hat{f}(w) = \frac{d}{dw}\int_{-\infty}^{\infty} \frac{e^{iwt} - 1}{it} f(t)\,dt \qquad \text{a.e.} \tag{1.13}$$

and

$$f(x) = \frac{d}{dx}\int_{-\infty}^{\infty} \frac{e^{-iwx} - 1}{-iw} \hat{f}(w)\,dw \qquad \text{a.e.} \tag{1.14}$$

A very elementary proof of the fact that (1.13) and (1.14) are eqivalent to (1.11) and (1.12), respectively, is given by Akhiezer and Glazman in their work on the theory of linear operators in Hilbert space [3, pp. 75–76]. These concepts of the Fourier transform were further developed by Titchmarch [99] for L^p functions $1 < p \leq 2$.

Theorem 5. Titchmarch's Theorem. Let $f \in L^p(\mathbb{R})$, $1 < p \leq 2$. Then there exists a function $\hat{f}(\xi) \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\left\| \hat{f}(\xi) - \int_{-N}^{N} f(t) e^{it\xi} dt \right\|_q \to 0 \qquad \text{as } N \to \infty$$

Furthermore

$$\left\| f(x) - \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(\xi) e^{-i\xi x} d\xi \right\|_p \to 0 \qquad \text{as } N \to \infty.$$

The Fourier receprocity relation also holds in the sense that

$$\hat{f}(\xi) = \frac{d}{d\xi} \int_{-\infty}^{\infty} f(t) \frac{e^{i\xi t} - 1}{it} dt \qquad \text{a.e.}$$

$$f(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{e^{-i\xi x} - 1}{-i\xi} d\xi \qquad \text{a.e.}$$

Also

$$\|\hat{f}\|_q \leq K(p) \left\{ \int_{-\infty}^{\infty} |f(x)|^p dx \right\}^{1/(p-1)},$$

where $K(p)$ is a constant depending upon p. Thus the Fourier integral operator $\mathcal{F}$ is a bounded linear operator from L^p to L^q. The work of generalizing the Fourier transform of functions was continued by Laurent Schwartz [87] who put forward the theory of the Fourier transform of tempered distributions and L. Ehrenpreis [36] who brought forward his theory of the Fourier transform of Schwartz distributions.

1.3. CONVOLUTION OF FUNCTIONS

Let f and g be complex-valued functions defined on the real line, which we denote by $\mathbb{R}$. Then their convolution $(f * g)(x)$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y)\, dy$$

provided that the above integral exists. At the set of points where the convolution exists, we are able to define a new function $(f * g)(x)$. Since many of the properties of the convolution defined above are similar to the product, we call the convolution

$(f * g)(x)$ a *convolution product of two functions* f and g. It is a simple exercise to show that $(f * g)(x) = (g * f)(x)$. I will now give some results in the form of theorems that demonstrate the existence of the convolution. The proof of the following theorem is found in Rudin [84, pp. 146–147].

Theorem 6. Let $f, g \in L^1(-\infty, \infty)$. Then

i. $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$ exists and is finite a.e.
ii. The function $(f * g)(x) \in L^1(-\infty, \infty)$.
iii. $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$,

where

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\,dx$$

Proof. There is no loss of generality in assuming that f and g are Borel measurable. Clearly, if f and g are Lebesgue measurable, then there exist Borel measurable functions f_0 and g_0, respectively, defined on the real line such that $f = f_0$ a.e. and $g = g_0$ a.e. Borel measurable functions are necessarily Lebesgue measurable, so we may assume that f and g are Borel measurable functions. Also the value of an integral remains unchanged by changing the values of the integrand at a set of points of measure zero. Now define

$$F(x, y) = f(x-y)g(y)$$

We want to first show that the function $F(x, y)$ is a Borel function in $\mathbb{R}^2$. For a set $E \in \mathbb{R}$, let there be a set $\tilde{E} \in \mathbb{R}^2$ defined by

$$\tilde{E} = \{(x, y) : x - y \in E\}$$

Since $x - y$ is a continuous function of (x, y), $\tilde{E}$ must be open whenever E is. It is very easy to verify that the collection of all $E \in \mathbb{R}$ for which $\tilde{E}$ (as defined above) is a Borel set forms a σ-algebra on $\mathbb{R}$. Again, if V is an open set in $\mathbb{R}$ and f is a Borel function on $\mathbb{R}$, the set $E = \{x : f(x) \in V\}$ is a Borel set in R. Therefore

$$\{(x, y) : f(x-y) \in V\} = \{(x, y) : x - y \in E\} = \tilde{E}$$

is a Borel set in $\mathbb{R}^2$. Hence the function $(x - y) \to f(x - y)$ is a Borel function. The function $(x, y) \to g(y)$ is also a Borel function in $\mathbb{R}^2$. Therefore the product $f(x - y)g(y)$ is a Borel function on $\mathbb{R}^2$. Now

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x, y)|\,dx = \int_{-\infty}^{\infty} |g(y)|\,dy \int_{-\infty}^{\infty} |f(x-y)|\,dx$$
$$= \|f\|_1 \, \|g\|_1$$

We have used the translation invariance property of the Lebesgue measure to show that

$$\int_{-\infty}^{\infty} |f(x-y)|dx = \|f\|_1$$

So $F(x, y) \in L^1(\mathbb{R}^2)$. Therefore in view of the Fubini's theorem $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$ exists for almost all $x \in \mathbb{R}$, and $(f * g)(x) \in L^1(\mathbb{R})$. This proves (i) and (ii) together. Now

$$\begin{aligned}\|f * g\|_1 &= \int_{-\infty}^{\infty} |(f * g)(x)|\,dx \leq \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |F(x, y)|\,dy\right] dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x, y)|\,dx\right) dy = \|f\|_1 \, \|g\|_1\end{aligned}$$

This proves (iii). □

Corollary 5. Let $f, g \in L^1(\mathbb{R})$. Then

$$\mathcal{F}(f * g)(w) = (\mathcal{F}f)(w)(\mathcal{F}g)(w)$$

where F is the Fourier transformation operator.

Proof.

$$\mathcal{F}(f * g)(w) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right) e^{iwx}dx$$

Then by Fubini's theorem we get

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)e^{iwx}dx\right) g(y)\,dy$$

Now

$$\begin{aligned}\int_{-\infty}^{\infty} f(x-y)e^{iwx}dx &= e^{iwy} \int_{-\infty}^{\infty} f(x-y)e^{iw(x-y)}dx \\ &= e^{iwy} \int_{-\infty}^{\infty} f(t)e^{iwt}dt\end{aligned}$$

By the translation invariance property of the Lebesgue measure,

$$\begin{aligned}\mathcal{F}(f * g)(w) &= \int_{-\infty}^{\infty} e^{iwy} \int_{-\infty}^{\infty} f(t)e^{iwt}dt\, g(y)\,dy \\ &= \int_{-\infty}^{\infty} g(y)e^{iwy}dy \int_{-\infty}^{\infty} f(t)e^{iwt}dt \\ &= (\mathcal{F}f)(w)(\mathcal{F}g)(w)\end{aligned}$$

An excellent proof of the following theorem is given by Hewitt and Stromberg [48, p. 397]. □

Theorem 7. For $1 < p < \infty$, let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Then for almost all $x \in \mathbb{R}$, $f(x-y)g(y)$ and $f(y)g(x-y)$ as functions of y, $\in L^1(\mathbb{R})$. For all such x define

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$$

and

$$(g * f)(x) = \int_{\mathbb{R}} g(x-y)f(y)\,dy$$

Then

$$(f * g)(x) = (g * f)(x) \qquad \text{a.e.}$$

and

$$(f * g)(x) \in L^p(\mathbb{R})$$

further $\|f * g\|_p \leq \|f\|_1 \, \|g\|_p$.

Proof. Let $q = \frac{p}{p-1}$, and let $h \in L^q(\mathbb{R})$. Then each of the functions $f(x-y)$, $g(y)$, $h(x)$, are Borel measurable in $\mathbb{R}^2$, and so also are their products taken two at a time and the function $f(x-y)\,g(y)\,h(x)$. Now using Fubini's theorem, translation invariance of the Lebesgue measure and Holder's inequality we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)g(y)h(x)|\,dy\,dx \\
&= \int_{-\infty}^{\infty} |h(x)| \int_{-\infty}^{\infty} |f(x-y)g(y)|\,dy\,dx \\
&= \int_{-\infty}^{\infty} |h(x)| \int_{-\infty}^{\infty} |f(t)g(x-t)|\,dt\,dx \\
&= \int_{-\infty}^{\infty} |f(t)| \int_{-\infty}^{\infty} |g(x-t)h(x)|\,dx\,dt \\
&\leq \int_{-\infty}^{\infty} |f(t)| \, \|g(x-t)\|_p \, \|h(x)\|_q \,dt \\
&= \|g\|_p \, \|h\|_q \, \|f\|_1 < \infty
\end{aligned}$$