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Financial Risk Manager Handbook

Fifth Edition

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Financial Risk Manager Handbook

Fifth Edition

PHILIPPE JORION
GARP

WILEY
John Wiley & Sons, Inc.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Jorion, Philippe, 1955–
Financial risk manager handbook / Philippe Jorion – 5th ed.
p. cm. – (Wiley finance series)
Includes index.
ISBN 978-0-470-47961-2 (paper/CD-ROM)
1. Financial risk management. 2. Risk management. 3. Corporate–Finance. I. Title.
HD61.J67 2009
332.64'5–dc22

2009008330

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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Preface

The *Financial Risk Manager Handbook* provides the core body of knowledge for financial risk managers. Risk management has evolved rapidly over the past decade and has become an indispensable function in many institutions.

This *Handbook* was originally written to provide support for candidates taking the FRM examination administered by GARP. As such, it reviews a wide variety of practical topics in a consistent and systematic fashion. It covers quantitative methods and capital markets, as well as market, credit, operational, and integrated risk management. It also discusses regulatory and legal issues essential to risk professionals.

This edition has been thoroughly updated to reflect recent developments in financial markets. The unprecedented losses incurred by many institutions have raised questions about risk management practices. These issues are now addressed in various parts of the book, which also include lessons from recent regulatory reports. The securitization process and structured credit products are critically examined. A new chapter on liquidity risk has been added, given the importance of this risk during the recent crisis. Finally, this *Handbook* incorporates the latest questions from the FRM examinations.

Modern risk management systems cut across the entire organization. This breadth is reflected in the subjects covered in this *Handbook*. The book was designed to be self-contained, but only for readers who already have some exposure to financial markets. To reap maximum benefit from this book, readers should have taken the equivalent of an MBA-level class on investments.

Finally, I want to acknowledge the help received in writing this *Handbook*. In particular, I thank the numerous readers who shared comments on previous editions. Any comment or suggestion for improvement will be welcome. This feedback will help us to maintain the high quality of the FRM designation.

Philippe Jorion
February 2009

About the Author

Philippe Jorion is a Professor of Finance at the Paul Merage School of Business at the University of California at Irvine. He has also taught at Columbia University, Northwestern University, the University of Chicago, and the University of British Columbia. He holds an M.B.A. and a Ph.D. from the University of Chicago and a degree in engineering from the University of Brussels. He is also a managing director at Pacific Alternative Asset Management Company (PAAMCO), a global fund of hedge funds.

Dr. Jorion is the author of more than 90 publications directed to academics and practitioners on the topics of risk management and international finance. He has also written a number of books, including *Big Bets Gone Bad: Derivatives and Bankruptcy in Orange County*, the first account of the largest municipal failure in U.S. history, and *Value at Risk: The New Benchmark for Managing Financial Risk*, which is aimed at finance practitioners and has become an industry standard.

Philippe Jorion is a frequent speaker at academic and professional conferences. He is on the editorial board of a number of finance journals and was editor in chief of the *Journal of Risk*.

About GARP

Founded in 1996, the Global Association of Risk Professionals (GARP) is the leading not-for-profit association for world-class financial risk certification, education, and training with close to 100,000 members representing 167 countries. With deep expertise and a strong reputation, GARP sets global standards and creates risk management programs valued worldwide. All GARP programs are developed with input from experts around the world to ensure that concepts and content reflect globally accepted practices.

GARP is dedicated to advancing the risk profession. For more information about GARP, please visit www.garp.com.

FINANCIAL RISK MANAGER (FRM®) CERTIFICATION

The benchmark FRM designation is the globally accepted risk management certification for financial risk professionals. The FRM objectively measures competency in the risk management profession based on globally accepted standards. With a compound annual growth rate of 25 percent over the past seven years, the FRM program has experienced significant growth in every financial center around the world. Now 16,000+ individuals hold the FRM designation in over 90 countries. In addition, organizations with five or more FRM registrants grew from 105 in 2003 to 424 in 2008, further demonstrating the FRM program's global acceptance.

The FRM Continuing Professional Education (CPE) program, to be offered starting in 2009 exclusively for certified FRM holders, provides the perspective and framework needed to further develop competencies in the ever-evolving field of risk management.

For more information about the FRM program, please visit www.garp.com/frmexam.

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The ICBRR allows individuals to expand their knowledge and understanding of the various risks, regulations, and supervisory requirements banks must face in today's economy, with emphasis on the Basel II Accord. This certificate is ideal for employees who are not professional risk managers but who have a strong need to understand risk concepts. The ICBRR program is designed for employees in nonrisk departments such as internal audit, accounting, information technology (IT), legal, compliance, and sales, acknowledging that everyone in the organization is a risk manager!

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The Certificate in Energy Risk Management provides individuals with a comprehensive and cross-product understanding of the physical and financial marketplaces relating to crude oil, natural gas, liquefied natural gas, and electricity/power. This program is valuable for anyone working in or servicing the energy field and requiring an understanding of the physical and financial markets, how they interrelate, and the risks involved. This program will launch in 2Q 2009.

Certificate in Risk Management for Islamic Financial Institutions

This certificate is under development by a practice oversight committee of Islamic finance experts from around the globe. The program will cover the risk management methodologies specific to Sharia'a-compliant financial products and will be the only one of its kind anywhere in the world.

GARP DIGITAL LIBRARY

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For more information, please visit www.garpdigitallibrary.org.

GARP EVENTS AND NETWORKING

GARP hosts major conventions throughout the world, where risk professionals come together to share knowledge, network, and learn from leading experts in the field. Conventions are bookended with interactive workshops that provide practical insights and case studies presented by the industry's leading practitioners.

GARP regional chapters provide an opportunity for financial risk professionals to network and share new trends and discoveries in risk management. Each one of our 52 chapters holds several meetings each year, in some locations more often, focusing on issues of importance to the risk management community, either globally or locally.

Introduction

GARP's formal mission is to be the leading professional association for financial risk managers, managed by and for its members and dedicated to the advancement of the risk profession through education, training, and the promotion of best practices globally. As a part of delivering on that mission, GARP has again teamed with Philippe Jorion to produce the fifth edition of the *Financial Risk Manager Handbook*.

The *Handbook* follows GARP's FRM Committee's published FRM Study Guide, which sets forth primary topics and subtopics covered in the FRM exam. The topics are selected by the FRM Committee as being representative of the theories and concepts utilized by risk management professionals as they address current issues.

Over the years the Study Guide has taken on an importance far exceeding its initial intent of providing guidance for FRM candidates. The Study Guide is now being used by universities, educators, and executives around the world to develop graduate-level business and finance courses, as a reference list for purchasing new readings for personal and professional libraries, as an objective outline to assess the risk management qualifications of an employee or a job applicant, and as guidance on the important trends currently affecting the financial risk management profession.

Given the expanded and dramatically growing recognition of the financial risk management profession globally, the *Handbook* has similarly assumed a natural and advanced role beyond its original purpose. It has now become the primary reference manual for risk professionals, academicians, and executives around the world. Professional risk managers must be well versed in a wide variety of risk-related concepts and theories, and must also keep themselves up-to-date with a rapidly changing marketplace. The *Handbook* is designed to allow them to do just that. It provides a financial risk management practitioner with the latest thinking and approaches to financial risk-related issues. It also provides coverage of advanced topics with questions and tutorials to enhance the reader's learning experience.

This fifth edition of the *Handbook* includes revised coverage of the primary topic areas covered by the FRM examination. Importantly, this edition also includes the latest lessons from the recent credit crisis, as well as new and more recent sample FRM questions.

The *Handbook* continues to keep pace with the dynamic financial risk profession while simultaneously offering serious risk professionals an excellent and cost-effective tool to keep abreast of the latest issues affecting the global risk management community.

Developing credibility and global acceptance for a professional certification program is a lengthy and complicated process. When GARP first administered its FRM exam in 1997, the concept of a professional risk manager and a global certification relating to that person's skill set was more theory than reality. That has now completely changed, as the number of current FRM holders exceeds 16,000.

The FRM is now the benchmark for a financial risk manager anywhere in the world. Professional risk managers having earned the FRM credential are globally recognized as having achieved a level of professional competency and a demonstrated ability to dynamically measure and manage financial risk in a real-world setting in accordance with global standards.

GARP is proud to continue to make this *Handbook* available to financial risk professionals around the world. Philippe Jorion, a preeminent risk management professional, has again compiled an exceptional reference book. Supplemented by an interactive test question CD, this *Handbook* is a requirement for any risk professional's library.

Global Association of Risk Professionals
February 2009

PART One

Quantitative Analysis

CHAPTER 1

Bond Fundamentals

Risk management starts with the pricing of assets. The simplest assets to study are regular, fixed-coupon bonds. Because their cash flows are predetermined, we can translate their stream of cash flows into a present value by discounting at a fixed interest rate. Thus the valuation of bonds involves understanding compounded interest, discounting, as well as the relationship between present values and interest rates.

Risk management goes one step further than pricing, however. It examines potential changes in the price of assets as the interest rate changes. In this chapter, we assume that there is a single interest rate, or yield, that is used to price the bond. This will be our fundamental risk factor. This chapter describes the relationship between bond prices and yields and presents indispensable tools for the management of fixed-income portfolios.

This chapter starts our coverage of quantitative analysis by discussing bond fundamentals. Section 1.1 reviews the concepts of discounting, present values, and future values. Section 1.2 then plunges into the price-yield relationship. It shows how the Taylor expansion rule can be used to relate movements in bond prices to those in yields. This Taylor expansion rule, however, covers much more than bonds. It is a building block of risk measurement methods based on local valuation, as we shall see later. Section 1.3 then presents an economic interpretation of duration and convexity.

The reader should be forewarned that this chapter, like many others in this handbook, is rather compact. This chapter provides a quick review of bond fundamentals with particular attention to risk measurement applications. By the end of this chapter, however, the reader should be able to answer advanced FRM questions on bond mathematics.

1.1 DISCOUNTING, PRESENT, AND FUTURE VALUE

An investor considers a zero-coupon bond that pays $100 in 10 years. Assume that the investment is guaranteed by the U.S. government, and that there is no credit risk. So, this is a default-free bond, which is exposed to market risk only. Because the payment occurs at a future date, the current value of the investment is surely less than an up-front payment of $100.

To value the payment, we need a **discounting factor**. This is also the **interest rate**, or more simply the **yield**. Define C_t as the cash flow at time t and the discounting factor as y. We define T as the number of periods until maturity, e.g., number of years, also known as **tenor**. The **present value** (PV) of the bond can be computed as

$$PV = \frac{C_T}{(1+y)^T} \tag{1.1}$$

For instance, a payment of $C_T = \$100$ in 10 years discounted at 6 percent is only worth \$55.84 now. So, all else fixed, the market value of zero-coupon bonds decreases with longer maturities. Also, keeping T fixed, the value of the bond decreases as the yield increases.

Conversely, we can compute the **future value** (FV) of the bond as

$$FV = PV \times (1+y)^T \tag{1.2}$$

For instance, an investment now worth $PV = \$100$ growing at 6 percent will have a future value of $FV = \$179.08$ in 10 years.

Here, the yield has a useful interpretation, which is that of an **internal rate of return** on the bond, or annual growth rate. It is easier to deal with rates of returns than with dollar values. Rates of return, when expressed in percentage terms and on an annual basis, are directly comparable across assets. An annualized yield is sometimes defined as the **effective annual rate (EAR)**.

It is important to note that the interest rate should be stated along with the method used for compounding. Annual compounding is very common. Other conventions exist, however. For instance, the U.S. Treasury market uses semiannual compounding. Define in this case y^S as the rate based on semiannual compounding. To maintain comparability, it is expressed in annualized form, i.e., after multiplication by 2. The number of periods, or semesters, is now $2T$. The formula for finding y^S is

$$PV = \frac{C_T}{(1+y^S/2)^{2T}} \tag{1.3}$$

For instance, a Treasury zero-coupon bond with a maturity of $T = 10$ years would have $2T = 20$ semiannual compounding periods. Comparing with (1.1), we see that

$$(1+y) = (1+y^S/2)^2 \tag{1.4}$$

Continuous compounding is often used when modeling derivatives. It is the limit of the case where the number of compounding periods per year increases to infinity. The continuously compounded interest rate y^C is derived from

$$PV = C_T \times e^{-y^C T} \tag{1.5}$$

where $e^{(\cdot)}$, sometimes noted as $\exp(\cdot)$, represents the exponential function.

Note that in all of these Equations (1.1), (1.3), and (1.5), the present value and future cash flows are identical. Because of different compounding periods, however, the yields will differ. Hence, the compounding period should always be stated.

Example: Using Different Discounting Methods

Consider a bond that pays \$100 in 10 years and has a present value of \$55.8395. This corresponds to an annually compounded rate of 6.00% using $PV = C_T/(1+y)^{10}$, or $(1+y) = (C_T/PV)^{1/10}$.

This rate can be transformed into a semiannual compounded rate, using $(1 + y^S/2)^2 = (1+y)$, or $y^S/2 = (1+y)^{1/2} - 1$, or $y^S = ((1+0.06)^{(1/2)} - 1) \times 2 = 0.0591 = 5.91\%$. It can be also transformed into a continuously compounded rate, using $\exp(y^C) = (1+y)$, or $y^C = \ln(1+0.06) = 0.0583 = 5.83\%$.

Note that as we increase the frequency of the compounding, the resulting rate decreases. Intuitively, because our money works harder with more frequent compounding, a lower investment rate will achieve the same payoff at the end.

KEY CONCEPT

For fixed present value and cash flows, increasing the frequency of the compounding will decrease the associated yield.

EXAMPLE 1.1: FRM EXAM 2002—QUESTION 48

An investor buys a Treasury bill maturing in 1 month for \$987. On the maturity date the investor collects \$1,000. Calculate effective annual rate (EAR).

a. 17.0%
b. 15.8%
c. 13.0%
d. 11.6%

EXAMPLE 1.2: FRM EXAM 2002—QUESTION 51

Consider a savings account that pays an annual interest rate of 8%. Calculate the amount of time it would take to double your money. Round to the nearest year.

a. 7 years
b. 8 years
c. 9 years
d. 10 years

1.2 PRICE–YIELD RELATIONSHIP

1.2.1 Valuation

The fundamental discounting relationship from Equation (1.1) can be extended to any bond with a fixed cash-flow pattern. We can write the present value of a bond P as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{1.6}$$

where: C_t = the cash flow (coupon or principal) in period t
t = the number of periods (e.g., half-years) to each payment
T = the number of periods to final maturity
y = the discounting factor per period (e.g., $y^S/2$)

A typical cash-flow pattern consists of a fixed coupon payment plus the repayment of the principal, or **face value** at expiration. Define c as the coupon *rate* and F as the face value. We have $C_t = cF$ prior to expiration, and at expiration, we have $C_T = cF + F$. The appendix reviews useful formulas that provide closed-form solutions for such bonds.

When the coupon rate c precisely matches the yield y, using the same compounding frequency, the present value of the bond must be equal to the face value. The bond is said to be a **par bond**. If the coupon is greater than the yield, the price must be greater than the face value, which means that this is a **premium bond**. Conversely, if the coupon is lower, or even zero for a zero-coupon bond, the price must be less than the face value, which means that this is a **discount bond**.

Equation (1.6) describes the relationship between the yield y and the value of the bond P, given its cash-flow characteristics. In other words, the value P can also be written as a nonlinear function of the yield y:

$$P = f(y) \tag{1.7}$$

Conversely, we can set P to the current market price of the bond, including any accrued interest. From this, we can compute the "implied" yield that will solve this equation.

Figure 1.1 describes the price-yield function for a 10-year bond with a 6% annual coupon. In risk management terms, this is also the relationship between the payoff on the asset and the risk factor. At a yield of 6%, the price is at par, $P = \$100$. Higher yields imply lower prices. This is an example of a **payoff function**, which links the price to the underlying risk factor.

Over a wide range of yield values, this is a highly nonlinear relationship. For instance, when the yield is zero, the value of the bond is simply the sum of cash

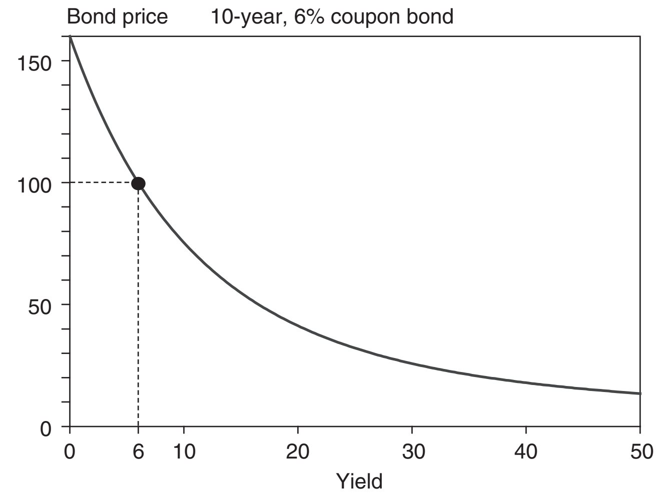

FIGURE 1.1 Price–Yield Relationship

flows, or \$160 in this case. When the yield tends to very large values, the bond price tends to zero. For small movements around the initial yield of 6%, however, the relationship is quasilinear.

There is a particularly simple relationship for **consols**, or **perpetual bonds**, which are bonds making regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed percentage of the face value, $C_t = C = cF$. As a result, the price can be simplified from Equation (1.6) to

$$P = cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] = \frac{c}{y}F \tag{1.8}$$

as shown in the appendix. In this case, the price is simply proportional to the inverse of the yield. Higher yields lead to lower bond prices, and vice versa.

Example: Valuing a Bond

Consider a bond that pays \$100 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 6%? If it falls to 5%?

The bond cash flows are $C_1 = \$6$, $C_2 = \$6, \ldots, C_{10} = \$106$. Using Equation (1.6) and discounting at 6%, this gives the present value of cash flows of \$5.66, \$5.34, ..., \$59.19, for a total of \$100.00. The bond is selling at par. This is logical because the coupon is equal to the yield, which is also annually compounded. Alternatively, discounting at 5% leads to a price of \$107.72.

1.2.2 Taylor Expansion

Let us say that we want to see what happens to the price if the yield changes from its initial value, called y_0, to a new value, $y_1 = y_0 + \Delta y$. Risk management is all about assessing the effect of changes in risk factors such as yields on asset values. Are there shortcuts to help us with this?

We could recompute the new value of the bond as $P_1 = f(y_1)$. If the change is not too large, however, we can apply a very useful shortcut. The nonlinear relationship can be approximated by a **Taylor expansion** around its initial value[1]

$$P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2} f''(y_0)(\Delta y)^2 + \cdots \tag{1.9}$$

where $f'(\cdot) = \frac{dP}{dy}$ is the first derivative and $f''(\cdot) = \frac{d^2P}{dy^2}$ is the second derivative of the function $f(\cdot)$ valued at the starting point.[2] This expansion can be generalized to situations where the function depends on two or more variables. For bonds, the first derivative is related to the *duration* measure, and the second to *convexity*.

Equation (1.9) represents an infinite expansion with increasing powers of Δy. Only the first two terms (linear and quadratic) are ever used by finance practitioners. They provide a good approximation to changes in prices relative to other assumptions we have to make about pricing assets. If the increment is very small, even the quadratic term will be negligible.

Equation (1.9) is fundamental for risk management. It is used, sometimes in different guises, across a variety of financial markets. We will see later that this Taylor expansion is also used to approximate the movement in the value of a derivatives contract, such as an option on a stock. In this case, Equation (1.9) is

$$\Delta P = f'(S)\Delta S + \frac{1}{2} f''(S)(\Delta S)^2 + \cdots \tag{1.10}$$

where S is now the price of the underlying asset, such as the stock. Here, the first derivative $f'(S)$ is called *delta*, and the second $f''(S)$, *gamma*.

The Taylor expansion allows easy aggregation across financial instruments. If we have x_i units (numbers) of bond i and a total of N different bonds in the portfolio, the portfolio derivatives are given by

$$f'(y) = \sum_{i=1}^{N} x_i f_i'(y) \tag{1.11}$$

[1] This is named after the English mathematician Brook Taylor (1685–1731), who published this result in 1715. The full recognition of the importance of this result only came in 1755 when Euler applied it to differential calculus.

[2] This first assumes that the function can be written in polynomial form as $P(y + \Delta y) = a_0 + a_1\Delta y + a_2(\Delta y)^2 + \cdots$, with unknown coefficients a_0, a_1, a_2. To solve for the first, we set $\Delta y = 0$. This gives $a_0 = P_0$. Next, we take the derivative of both sides and set $\Delta y = 0$. This gives $a_1 = f'(y_0)$. The next step gives $2a_2 = f''(y_0)$. Here, the term "derivatives" takes the usual mathematical interpretation, and has nothing to do with *derivatives products* such as options.

1.3 BOND PRICE DERIVATIVES

For fixed-income instruments, the derivatives are so important that they have been given a special name.[3] The negative of the first derivative is the **dollar duration (DD)**:

$$f'(y_0) = \frac{dP}{dy} = -D^* \times P_0 \tag{1.12}$$

where D^* is called the **modified duration**. Thus, dollar duration is

$$\mathrm{DD} = D^* \times P_0 \tag{1.13}$$

where the price P_0 represent the *market* price, including any accrued interest. Sometimes, risk is measured as the **dollar value of a basis point (DVBP)**,

$$\mathrm{DVBP} = \mathrm{DD} \times \Delta y = [D^* \times P_0] \times 0.0001 \tag{1.14}$$

with 0.0001 representing an interest rate change of one basis point (bp) or one hundredth of a percent. The **DVBP**, sometimes called the **DV01**, measures can be easily added up across the portfolio.

The second derivative is the **dollar convexity (DC)**:

$$f''(y_0) = \frac{d^2P}{dy^2} = C \times P_0 \tag{1.15}$$

where C is called the **convexity**.

For fixed-income instruments with known cash flows, the price-yield function is known, and we can compute analytical first and second derivatives. Consider, for example, our simple zero-coupon bond in Equation (1.1) where the only payment is the face value, $C_T = F$. We take the first derivative, which is

$$\frac{dP}{dy} = \frac{d}{dy}[\frac{F}{(1+y)^T}] = (-T)\frac{F}{(1+y)^{T+1}} = -\frac{T}{(1+y)}P \tag{1.16}$$

Comparing with Equation (1.12), we see that the modified duration must be given by $D^* = T/(1+y)$. The conventional measure of **duration** is $D = T$, which does not include division by $(1+y)$ in the denominator. This is also called **Macaulay duration**. Note that duration is expressed in periods, like T. With annual compounding, duration is in years. With semiannual compounding, duration is in semesters. It then has to be divided by two for conversion to years. Modified

[3] Note that this chapter does not present duration in the traditional textbook order. In line with the advanced focus on risk management, we first analyze the properties of duration as a sensitivity measure. This applies to any type of fixed-income instrument. Later, we will illustrate the usual definition of duration as a weighted average maturity, which applies for fixed-coupon bonds only.

duration D^* is related to Macaulay duration D

$$D^* = \frac{D}{(1+y)} \tag{1.17}$$

Modified duration is the appropriate measure of interest rate exposure. The quantity $(1+y)$ appears in the denominator because we took the derivative of the present value term with discrete compounding. If we use continuous compounding, modified duration is identical to the conventional duration measure. In practice, the difference between Macaulay and modified duration is usually small.

Let us now go back to Equation (1.16) and consider the second derivative, which is

$$\frac{d^2P}{dy^2} = -(T+1)(-T)\frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \times P \tag{1.18}$$

Comparing with Equation (1.15), we see that the convexity is $C = (T+1)T/(1+y)^2$. Note that its dimension is expressed in period squared. With semiannual compounding, convexity is measured in semesters squared. It then has to be divided by 4 for conversion to years squared.[4] So, convexity must be positive for bonds with fixed coupons.

Putting together all these equations, we get the Taylor expansion for the change in the price of a bond, which is

$$\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2 + \cdots \tag{1.19}$$

Therefore duration measures the first-order (linear) effect of changes in yield and convexity the second-order (quadratic) term.

Example: Computing the Price Approximation[5]

Consider a 10-year zero-coupon Treasury bond trading at a yield of 6 percent. The present value is obtained as $P = 100/(1 + 6/200)^{20} = 55.368$. As is the practice in the Treasury market, yields are semiannually compounded. Thus all computations should be carried out using semesters, after which final results can be converted into annual units.

Here, Macaulay duration is exactly 10 years, as $D = T$ for a zero coupon bond. Its modified duration is $D^* = 20/(1 + 6/200) = 19.42$ semesters, which is 9.71 years. Its convexity is $C = 21 \times 20/(1 + 6/200)^2 = 395.89$ semesters

[4] This is because the conversion to annual terms is obtained by multiplying the semiannual yield Δy by two. As a result, the duration term must be divided by 2 and the convexity term by 2^2, or 4, for conversion to annual units.

[5] For such examples in this handbook, please note that intermediate numbers are reported with fewer significant digits than actually used in the computations. As a result, using rounded off numbers may give results that differ slightly from the final numbers shown here.

squared, which is 98.97 in years squared. Dollar duration is $\mathrm{DD} = D^* \times P = 9.71 \times \$55.37 = \$537.55$. The DVBP is $\mathrm{DVBP} = \mathrm{DD} \times 0.0001 = \$0.0538$.

We want to approximate the change in the value of the bond if the yield goes to 7%. Using Equation (1.19), we have $\Delta P = -[9.71 \times \$55.37](0.01) + 0.5[98.97 \times \$55.37](0.01)^2 = -\$5.375 + \$0.274 = -\$5.101$. Using the linear term only, the new price is $\$55.368 - \$5.375 = \$49.992$. Using the two terms in the expansion, the predicted price is slightly higher, at $\$55.368 - \$5.375 + \$0.274 = \$50.266$.

These numbers can be compared with the exact value, which is \$50.257. The linear approximation has a relative pricing error of −0.53%, which is not bad. Adding a quadratic term reduces this to an error of 0.02% only, which is very small, given typical bid-ask spreads.

More generally, Figure 1.2 compares the quality of the Taylor series approximation. We consider a 10-year bond paying a 6 percent coupon semiannually. Initially, the yield is also at 6 percent and, as a result, the price of the bond is at par, at \$100. The graph compares three lines representing

1. The actual, exact price $\quad P = f(y_0 + \Delta y)$
2. The duration estimate $\quad P = P_0 - D^* P_0 \Delta y$
3. The duration and convexity estimate $\quad P = P_0 - D^* P_0 \Delta y + (1/2) C P_0 (\Delta y)^2$

The actual price curve shows an increase in the bond price if the yield falls and, conversely, a depreciation if the yield increases. This effect is captured by the tangent to the true price curve, which represents the linear approximation based on duration. For small movements in the yield, this linear approximation provides a reasonable fit to the exact price.

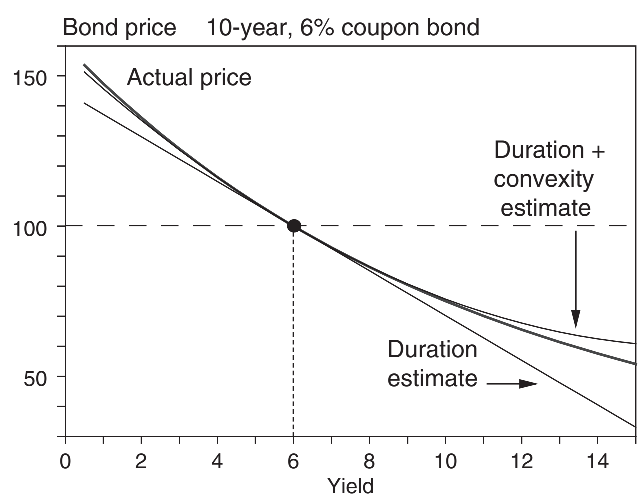

FIGURE 1.2 Price Approximation

KEY CONCEPT

Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point.

For large movements in price, however, the price-yield function becomes more curved and the linear fit deteriorates. Under these conditions, the quadratic approximation is noticeably better.

We should also note that the curvature is away from the origin, which explains the term *convexity* (as opposed to concavity). Figure 1.3 compares curves with different values for convexity. This curvature is beneficial since the second-order effect $0.5[C \times P](\Delta y)^2$ *must* be positive when convexity is positive.

As the figure shows, when the yield rises, the price drops but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than along the tangent. In other words, the quadratic term is always beneficial.

KEY CONCEPT

Convexity is always positive for regular coupon-paying bonds. Greater convexity is beneficial both for falling and rising yields.

The bond's modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity cannot be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are uncertain. Instead, the portfolio manager has access to pricing models that can be used to reprice the securities under various yield environments.

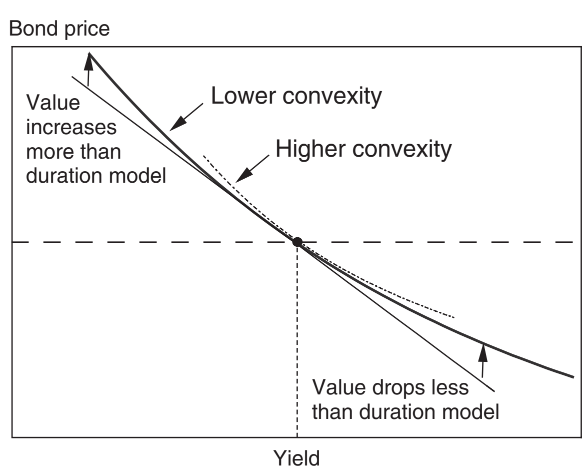

FIGURE 1.3 Effect of Convexity

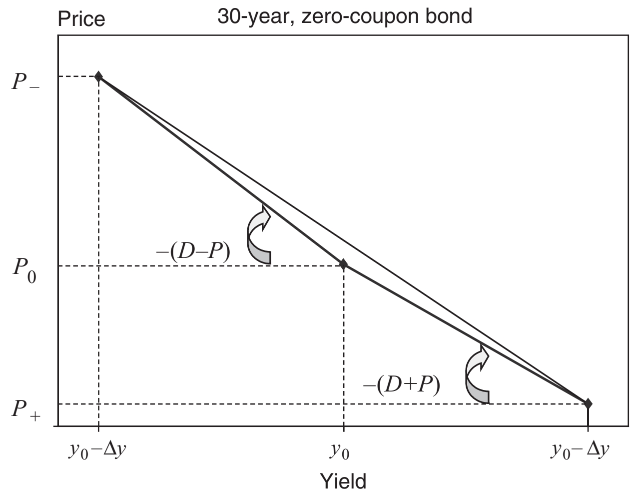

FIGURE 1.4 Effective Duration and Convexity

As shown in Figure 1.4, we choose a change in the yield, Δy, and reprice the bond under an upmove scenario, $P_+ = P(y_0 + \Delta y)$, and downmove scenario, $P_- = P(y_0 - \Delta y)$. **Effective duration** is measured by the numerical derivative. Using $D^* = -(1/P)dP/dy$, it is estimated as

$$D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2\Delta y)P_0} \tag{1.20}$$

Using $C = (1/P)d^2P/dy^2$, **effective convexity** is estimated as

$$C^E = [D_- - D_+]/\Delta y = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0\Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0\Delta y)}\right]/\Delta y \tag{1.21}$$

To illustrate, consider a 30-year zero-coupon bond with a yield of 6%, semiannually compounded. The initial price is \$16.9733. We revalue the bond at 5% and 7%, with prices shown in the table. The effective duration in Equation (1.20) uses the two extreme points. The effective convexity in Equation (1.21) uses the difference between the dollar durations for the upmove and downmove. Note that convexity is positive if duration increases as yields fall, or if $D_- > D_+$.

The computations are detailed in Table 1.1, which shows an effective duration of 29.56. This is very close to the true value of 29.13, and would be even closer if the step Δy was smaller. Similarly, the effective convexity is 869.11, which is close to the true value of 862.48.

Finally, this numerical approach can be applied to get an estimate of the duration of a bond by considering bonds with the same maturity but different coupons. If interest rates decrease by 1%, the market price of a 6% bond should go up to a value close to that of a 7% bond. Thus we replace a drop in yield of Δy with an increase in coupon Δc and use the effective duration method to find

TABLE 1.1 Effective Duration and Convexity

State	Yield (%)	Bond Value	Duration Computation	Convexity Computation
Initial y_0	6.00	16.9733		
Up $y_0 + \Delta y$	7.00	12.6934		Duration up: 25.22
Down $y_0 - \Delta y$	5.00	22.7284		Duration down: 33.91
Difference in values			−10.0349	8.69
Difference in yields			0.02	0.01
Effective measure			29.56	869.11
Exact measure			29.13	862.48

the **coupon curve duration**[6]

$$D^{CC} = \frac{[P_+ - P_-]}{(2P_0\Delta c)} = \frac{P(y_0; c + \Delta c) - P(y_0; c - \Delta c)}{(2\Delta c)P_0} \tag{1.22}$$

This approach is useful for securities which are difficult to price under various yield scenarios. It only requires the market prices of securities with different coupons.

Example: Computation of Coupon Curve Duration

Consider a 10-year bond that pays a 7% coupon semiannually. In a 7% yield environment, the bond is selling at par and has modified duration of 7.11 years. The prices of 6% and 8% coupon bonds are \$92.89 and \$107.11, respectively. This gives a coupon curve duration of $(107.11 - 92.89)/(0.02 \times 100) = 7.11$, which in this case is the same as modified duration.

EXAMPLE 1.3: FRM EXAM 2006—QUESTION 75

A zero-coupon bond with a maturity of 10 years has an annual effective yield of 10%. What is the closest value for its modified duration?

a. 9
b. 10
c. 99
d. 100

[6] For a more formal proof, we could take the pricing formula for a consol at par and compute the derivatives with respect to y and c. Apart from the sign, these derivatives are identical when $y = c$.

EXAMPLE 1.4: FRM EXAM 2007—QUESTION 115

A portfolio manager has a bond position worth USD 100 million. The position has a modified duration of eight years and a convexity of 150 years. Assume that the term structure is flat. By how much does the value of the position change if interest rates increase by 25 basis points?

a. USD −2,046,875
b. USD −2,187,500
c. USD −1,953,125
d. USD −1,906,250

EXAMPLE 1.5: FRM EXAM 2007—QUESTION 55

Consider the following three methods of estimating the profit and loss (P&L) of a bullet bond: full repricing, duration (PV01), and duration plus convexity. Rank the methods to estimate the P&L impact of a large negative yield shock from the lowest to the highest.

a. Duration, duration plus convexity, full repricing
b. Duration, full repricing, duration plus convexity
c. Duration plus convexity, duration, full repricing
d. Full repricing, duration plus convexity, duration

1.3.1 Interpreting Duration and Convexity

The preceding section has shown how to compute analytical formulas for duration and convexity in the case of a simple zero-coupon bond. We can use the same approach for coupon-paying bonds. Going back to Equation (1.6), we have

$$\frac{dP}{dy} = \sum_{t=1}^{T} \frac{-tC_t}{(1+y)^{t+1}} = -[\sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}]/P \times \frac{P}{(1+y)} = -\frac{D}{(1+y)}P \qquad (1.23)$$

which defines duration as

$$D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}/P \qquad (1.24)$$

The economic interpretation of duration is that it represents the average time to wait for each payment, weighted by the present value of the associated cash flow. Indeed, replacing P, we can write

$$D = \sum_{t=1}^{T} t \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} t \times w_t \tag{1.25}$$

where the weights w_t represent the ratio of the present value of each cash flow C_t relative to the total, and sum to unity. This explains why the duration of a zero-coupon bond is equal to the maturity. There is only one cash flow and its weight is one.

KEY CONCEPT

(Macaulay) duration represents an average of the time to wait for all cash flows.

Figure 1.5 lays out the present value of the cash flows of a 6% coupon, 10-year bond. Given a duration of 7.80 years, this coupon-paying bond is equivalent to a zero-coupon bond maturing in exactly 7.80 years.

For bonds with fixed coupons, duration is less than maturity. For instance, Figure 1.6 shows how the duration of a 10-year bond varies with its coupon. With a zero coupon, Macaulay duration is equal to maturity. Higher coupons place more weight on prior payments and therefore reduce duration.

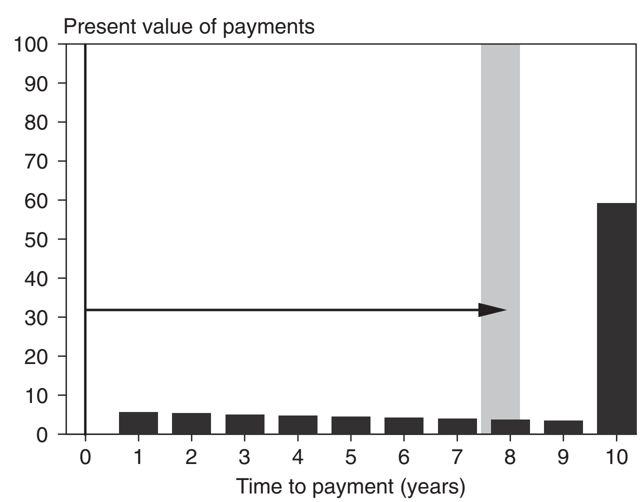

FIGURE 1.5 Duration as the Maturity of a Zero-Coupon Bond

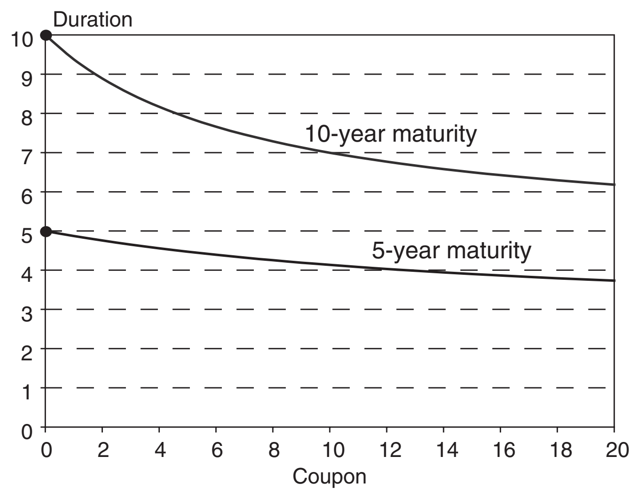

FIGURE 1.6 Duration and Coupon

Duration can be expressed in a simple form for **consols**. From Equation (1.8), we have $P = (c/y)F$. Taking the derivative, we find

$$\frac{dP}{dy} = cF\frac{(-1)}{y^2} = (-1)\frac{1}{y}[\frac{c}{y}F] = (-1)\frac{1}{y}P = -\frac{D_C}{(1+y)}P \tag{1.26}$$

Hence the Macaulay duration for the consol D_C is

$$D_C = \frac{(1+y)}{y} \tag{1.27}$$

This shows that the duration of a consol is finite even if its maturity is infinite. Also, this duration does not depend on the coupon.

This formula provides a useful rule of thumb. For a long-term coupon-paying bond, duration should be lower than $(1+y)/y$. For instance, when $y = 6\%$, the upper limit on duration is $D_C = 1.06/0.06$, or 17.7 years. In this environment, the duration of a par 30-year bond is 14.25, which is indeed lower than 17.7 years.

KEY CONCEPT

The duration of a long-term bond can be approximated by an upper bound, which is that of a consol with the same yield, $D_C = (1+y)/y$.

Figure 1.7 describes the relationship between duration, maturity, and coupon for regular bonds in a 6% yield environment. For the zero-coupon bond, $D = T$, which is a straight line going through the origin. For the par 6% bond, duration increases monotonically with maturity until it reaches the asymptote of D_C. The

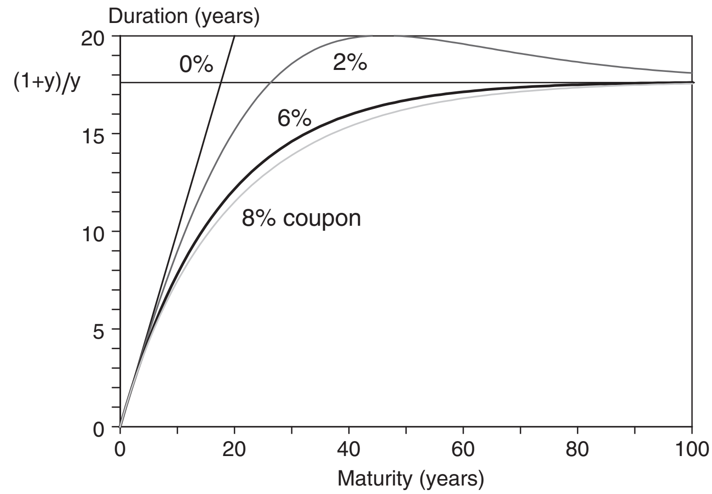

FIGURE 1.7 Duration and Maturity

8% bond has lower duration than the 6% bond for fixed T. Greater coupons, for a fixed maturity, decrease duration, as more of the payments come early.

Finally, the 2% bond displays a pattern intermediate between the zero-coupon and 6% bonds. It initially behaves like the zero, exceeding D_C initially then falling back to the asymptote, which is the same for all coupon-paying bonds.

Taking now the second derivative in Equation (1.23), we have

$$\frac{d^2 P}{dy^2} = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} = \left[\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P \right] \times P \tag{1.28}$$

which defines convexity as

$$C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P \tag{1.29}$$

Convexity can also be written as

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times w_t \tag{1.30}$$

Because the squared t term dominates in the fraction, this basically involves a weighted average of the square of time. Therefore, convexity is much greater for long-maturity bonds because they have payoffs associated with large values of t. The formula also shows that convexity is always positive for such bonds, implying that the curvature effect is beneficial. As we will see later, convexity can be negative for bonds that have uncertain cash flows, such as **mortgage-backed securities** (MBSs) or callable bonds.

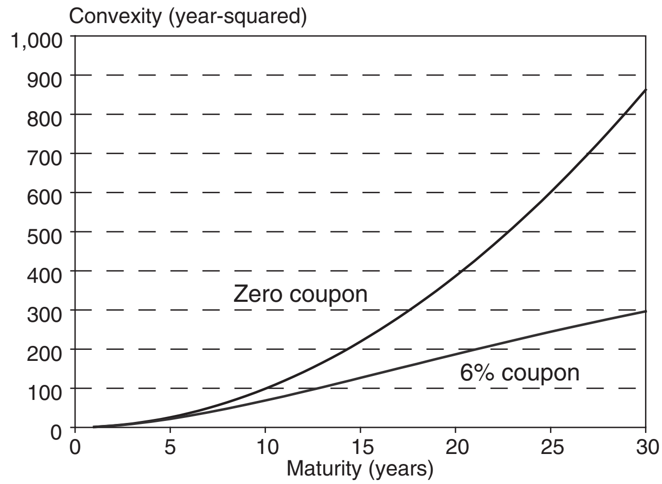

FIGURE 1.8 Convexity and Maturity

Figure 1.8 displays the behavior of convexity, comparing a zero-coupon bond with a 6% coupon bond with identical maturities. The zero-coupon bond always has greater convexity, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity, for example about 900 for the 30-year zero. In contrast, the 30-year coupon bond has a convexity of about 300 only.

KEY CONCEPT

All else equal, duration and convexity both increase for longer maturities, lower coupons, and lower yields.

As an illustration, Table 1.2 details the steps of the computation of duration and convexity for a two-year, 6% semiannual coupon-paying bond. We first convert the annual coupon and yield into semiannual equivalent, \$3 and 3% each. The PV column then reports the present value of each cash flow. We verify that these add up to \$100, since the bond must be selling at par.

Next, the duration term column multiplies each PV term by time, or more precisely the number of half years until payment. This adds up to \$382.86, which divided by the price gives $D = 3.83$. This number is measured in half years, and we need to divide by two to convert to years. Macaulay duration is 1.91 years, and modified duration $D^* = 1.91/1.03 = 1.86$ years. Note that, to be consistent, the adjustment in the denominator involves the semiannual yield of 3%.

Finally, the right-most column shows how to compute the bond's convexity. Each term involves PV_t times $t(t+1)/(1+y)^2$. These terms sum to 1,777.755, or divided by the price, 17.78. This number is expressed in units of time squared and must be divided by 4 to be converted in annual terms. We find a convexity of $C = 4.44$, in year-squared.

TABLE 1.2 Computing Duration and Convexity

Period (half-year) t	Payment C_t	Yield (%) (6 mo)	PV of Payment $C_t/(1+y)^t$	Duration Term tPV_t	Convexity Term $t(t+1)PV_t \times (1/(1+y)^2)$
1	3	3.00	2.913	2.913	5.491
2	3	3.00	2.828	5.656	15.993
3	3	3.00	2.745	8.236	31.054
4	103	3.00	91.514	366.057	1725.218
Sum:			100.00	382.861	1777.755
(half-years)				3.83	17.78
(years)				1.91	
Modified duration				1.86	
Convexity					4.44

EXAMPLE 1.6: FRM EXAM 2003—QUESTION 13

Suppose the face value of a three-year option-free bond is USD 1,000 and the annual coupon is 10%. The current yield to maturity is 5%. What is the modified duration of this bond?

a. 2.62
b. 2.85
c. 3.00
d. 2.75

EXAMPLE 1.7: FRM EXAM 2002—QUESTION 118

A Treasury bond has a coupon rate of 6% per annum (the coupons are paid semiannually) and a semiannually compounded yield of 4% per annum. The bond matures in 18 months and the next coupon will be paid 6 months from now. Which number below is closest to the bond's Macaulay duration?

a. 1.023 years
b. 1.457 years
c. 1.500 years
d. 2.915 years

EXAMPLE 1.8: DURATION AND COUPON

A and B are two perpetual bonds, that is, their maturities are infinite. A has a coupon of 4% and B has a coupon of 8%. Assuming that both are trading at the same yield, what can be said about the duration of these bonds?

a. The duration of A is greater than the duration of B.
b. The duration of A is less than the duration of B.
c. A and B both have the same duration.
d. None of the above.

EXAMPLE 1.9: FRM EXAM 2004—QUESTION 16

A manager wants to swap a bond for a bond with the same price but higher duration. Which of the following bond characteristics would be associated with a higher duration?

I. A higher coupon rate
II. More frequent coupon payments
III. A longer term to maturity
IV. A lower yield

a. I, II, and III
b. II, III, and IV
c. III and IV
d. I and II

EXAMPLE 1.10: FRM EXAM 2001—QUESTION 104

When the maturity of a plain coupon bond increases, its duration increases

a. Indefinitely and regularly
b. Up to a certain level
c. Indefinitely and progressively
d. In a way dependent on the bond being priced above or below par

EXAMPLE 1.11: FRM EXAM 2000—QUESTION 106

Consider the following bonds:

Bond Number	Maturity (yrs)	Coupon Rate	Frequency	Yield (Annual)
1	10	6%	1	6%
2	10	6%	2	6%
3	10	0%	1	6%
4	10	6%	1	5%
5	9	6%	1	6%

How would you rank the bonds from the shortest to longest duration?

a. 5-2-1-4-3
b. 1-2-3-4-5
c. 5-4-3-1-2
d. 2-4-5-1-3

EXAMPLE 1.12: FRM EXAM 2000—QUESTION 110

Which of the following statements are *true*?

I. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 10-year, 6% bond.
II. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 6% bond with a duration of 10 years.
III. Convexity grows proportionately with the maturity of the bond.
IV. Convexity is always positive for all types of bonds.
V. Convexity is always positive for “straight” bonds.

a. I only
b. I and II only
c. I and V only
d. II, III, and V only